

# Optimal Local and Remote Controllers with Unreliable Uplink Channels

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## Abstract

We consider a networked control system consisting of a remote controller and a collection of linear plants, each associated with a local controller. Each local controller directly observes the state of its co-located plant and can inform the remote controller of the plant's state through an unreliable uplink channel. We assume that the downlink channels from the remote controller to local controllers are perfect. The objective of the local controllers and the remote controller is to cooperatively minimize a quadratic performance cost. We provide a dynamic program for this decentralized control problem using the common information approach. Although our problem is not a partially nested problem, we obtain explicit optimal strategies for all controllers. In the optimal strategies, all controllers compute common estimates of the states of the plants based on the common information from the communication network. The remote controller's action is linear in the common estimated states, and the action of each local controller is linear in both the actual state of its co-located plant and the common estimated states. We illustrate our results with a platoon control problem for autonomous vehicles.

## I. INTRODUCTION

The advent of information and communication technologies along with the development of the Internet of Things (IoT) has drawn more attention to networked control systems (NCSs). NCSs are distributed systems in which information is exchanged through a network among various components (controllers, smart sensors, actuators, etc.). The connectivity of NCS brings numerous opportunities to new applications such as autonomous vehicles, smart grid, remote surgery,

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smart home, and large manufacturing systems (see [2–4] and references therein). However, the network connection is subjected to various communication constraints. One main constraint is the unreliability of communication channels which can greatly affect the performance of NCS [5], [6]. Therefore, the study of NCS over unreliable channels is of great importance.

The effect of control over unreliable channels has been investigated in [7–12] for NCS with a single controller. However, most NCS applications consist of multiple sub-systems where each sub-system may be controlled by a remote controller as well as a local controller. For example, in unmanned aerial vehicle (UAV) systems, the UAVs are remotely controlled by a ground control station while a local computer in each UAV provides basic stability controls [13]. The overall system performance depends on the coordination among the remote controller and all local controllers through the communication network. In this paper, we consider a NCS consisting of a remote controller and a collection of linear plants, each associated with a local controller as shown in Fig. 1. Each plant is directly controlled by a local controller which can perfectly observe the state of the plant. The remote controller can control all plants, but it does not have direct access to the states as its name suggests. The objective of the local controllers and the remote controller is to cooperatively minimize an overall quadratic performance cost of the NCS. The remote controller and local controllers are connected by a communication network where the downlinks from the remote controller to local controllers are perfect but the uplinks from local controllers to the remote controller are unreliable channels with random packet drops. Such scenario happens in many situations where the remote controller is equipped with sufficient communication resources, but each local controller has limited transmission capabilities. For instance, the local controllers can be a group of battery-powered telerobots or autonomous vehicles with limited transmission power proximal to their co-located systems while the remote controller can be a controlling operator connected to a power outlet or a base station with high transmission power.

When the local controllers are smart sensors or encoders that can only sense and transmit information, the NCS operation depends only on remote estimation and control. Remote estimation with a single smart sensor has been studied in [14–17] and has been extended to the case with multiple smart sensors and general packet drop models in [18], [19]. Remote estimation and control of a linear plant has been studied in [20–25] under various channel models between smart sensors and a remote controller. The problem considered in this paper is different from these previous works on NCS because our problem is a decentralized control problem with

multiple controllers where the dynamics of each plant is controlled by the remote controller as well as the corresponding local controller. Finding optimal strategies in decentralized control problems is generally considered a difficult problem (see [26–28]). In general, linear control strategies are not optimal, and even the problem of finding the best linear control strategies is not convex [29]. Existing optimal solutions of decentralized control problems require either specific information structures, such as partially nested [30–35], stochastically nested [36], or other specific properties, such as quadratic invariance [37] or substitutability [38], [39].

For the problem we consider in this paper, none of the above properties hold due to either the unreliable communication or the nature of dynamics and cost function. We use the common information approach to show that this problem is equivalent to a centralized sequential decision-making problem where the remote controller is the only decision-maker. We provide a dynamic program to obtain the optimal strategies of the remote controller in the equivalent problem. Then, using the optimal strategies of the equivalent problem, we obtain explicit optimal strategies for all local controllers and the remote controller. In the optimal strategies, all controllers compute common estimates of the states of the plants based on the common information from the communication network. The remote controller’s action is linear in the common estimated states, and the action of each local controller is linear in both the actual state of its co-located plant and the common estimated states. As an application of our problem, we apply our results to a simple platoon control problem for autonomous vehicles.

#### A. Notation

Random variables/vectors are denoted by upper case letters, their realization by the corresponding lower case letter. For a sequence of column vectors  $X, Y, Z, \dots$ , the notation  $\text{vec}(X, Y, Z, \dots)$  denotes the vector  $[X^\top, Y^\top, Z^\top, \dots]^\top$ . The transpose and trace of matrix  $A$  are denoted by  $A^\top$  and  $\text{tr}(A)$ , respectively. In general, subscripts are used as time index while superscripts are used to index controllers. For time indices  $t_1 \leq t_2$ ,  $X_{t_1:t_2}$  (resp.  $g_{t_1:t_2}(\cdot)$ ) is the shorthand notation for the variables  $X_{t_1}, X_{t_1+1}, \dots, X_{t_2}$  (resp. functions  $g_{t_1}(\cdot), \dots, g_{t_2}(\cdot)$ ). Similarly, for  $n_1 \leq n_2$ ,  $X^{n_1:n_2}$  (resp.  $g^{n_1:n_2}(\cdot)$ ) is the shorthand notation for the variables  $X^{n_1}, X^{n_1+1}, \dots, X^{n_2}$  (resp. functions  $g^{n_1}(\cdot), \dots, g^{n_2}(\cdot)$ ). For set  $\mathcal{A} = \{\alpha_1, \dots, \alpha_N\}$ , the collection  $X^{\alpha_1}, \dots, X^{\alpha_N}$  (resp.  $g^{\alpha_1}(\cdot), \dots, g^{\alpha_N}(\cdot)$ ) is denoted by  $\{X^m\}_{m \in \mathcal{A}}$  (resp.  $\{g^m(\cdot)\}_{m \in \mathcal{A}}$ ). Furthermore, the notation  $X^{-\alpha_n}$  is used to denote  $\{X^m\}_{m \in \mathcal{A} \setminus \{\alpha_n\}}$ . The intersection of the events  $E^{\alpha_1}, \dots, E^{\alpha_N}$  is denoted by

$\{E^m\}_{m \in \mathcal{A}}$ .

The indicator function of set  $E$  is denoted by  $\mathbb{1}_E(\cdot)$ , that is,  $\mathbb{1}_E(x) = 1$  if  $x \in E$ , and 0 otherwise. If  $E$  is an event, then  $\mathbb{1}_E$  denotes the resulting random variable.  $\mathbb{P}(\cdot)$ ,  $\mathbb{E}[\cdot]$ , and  $\text{cov}(\cdot)$  denote the probability of an event, the expectation of a random variable/vector, and the covariance matrix of a random vector, respectively. For random variables/vectors  $X$  and  $Y$ ,  $\mathbb{P}(\cdot|Y = y)$  denotes the probability of an event given that  $Y = y$ , and  $\mathbb{E}[X|y] := \mathbb{E}[X|Y = y]$ . For a strategy  $g$ , we use  $\mathbb{P}^g(\cdot)$  (resp.  $\mathbb{E}^g[\cdot]$ ) to indicate that the probability (resp. expectation) depends on the choice of  $g$ . Let  $\Delta(\mathbb{R}^n)$  denote the set of all probability measures on  $\mathbb{R}^n$  with finite second moment. For any  $\theta \in \Delta(\mathbb{R}^n)$ ,  $\theta(E) = \int_{\mathbb{R}^n} \mathbb{1}_E(x) \theta(dx)$  denotes the probability of event  $E$  under  $\theta$ . The mean and the covariance of a distribution  $\theta \in \Delta(\mathbb{R}^n)$  are denoted by  $\mu(\theta)$  and  $\text{cov}(\theta)$ , respectively, and are defined as  $\mu(\theta) = \int_{\mathbb{R}^n} x \theta(dx)$  and  $\text{cov}(\theta) = \int_{\mathbb{R}^n} (x - \mu(\theta))(x - \mu(\theta))^\top \theta(dx)$ .

The notation  $\mathbf{I}_n$  and  $\mathbf{0}_{n \times m}$  is used to denote a  $n \times n$  identity matrix and a  $n \times m$  zero matrix, respectively. For block matrix  $B$ ,  $[B]_{r,n}$  denotes the  $n$ -th block row of  $B$ . For example, for  $B = \begin{bmatrix} \mathbf{I}_m & \mathbf{0}_{m \times n} \\ \mathbf{0}_{n \times m} & \mathbf{I}_n \end{bmatrix}$ ,  $[B]_{r,1} = [\mathbf{I}_m \quad \mathbf{0}_{m \times n}]$  and  $[B]_{r,2} = [\mathbf{0}_{n \times m} \quad \mathbf{I}_n]$ .

## B. Organization

The rest of the paper is organized as follows. We introduce the system model and formulate the multi-controller NCS problem in Section II. In Section III, we formulate an equivalent problem using the common information approach and provide a dynamic program for this problem. We solve the dynamic program in Section IV. In Section V, we consider an application for autonomous vehicles using the system model of Section II. Section VI concludes the paper. The proofs of all the technical results of the paper appear in the Appendices.

## II. SYSTEM MODEL AND PROBLEM FORMULATION

Consider a discrete-time system with  $N$  plants,  $N$  local controllers,  $C^1, C^2, \dots, C^N$ , and one remote controller  $C^0$  as shown in Fig. 1. We use  $\mathcal{N}$  to denote the set  $\{1, 2, \dots, N\}$  and  $\bar{\mathcal{N}}$  to denote  $\{0, 1, \dots, N\}$ . The linear dynamics of plant  $n \in \mathcal{N}$  are given by

$$X_{t+1}^n = A^{nn} X_t^n + B^{nn} U_t^n + B^{n0} U_t^0 + W_t^n, t = 0, \dots, T \quad (1)$$

where  $X_t^n \in \mathbb{R}^{d_x^n}$  is the state of the plant  $n \in \mathcal{N}$  at time  $t$ ,  $U_t^n \in \mathbb{R}^{d_u^n}$  is the control action of the controller  $C^n$ ,  $n \in \bar{\mathcal{N}}$ , and  $A^{nn}, B^{nn}, B^{n0}$ ,  $n \in \mathcal{N}$ , are matrices with appropriate dimensions.

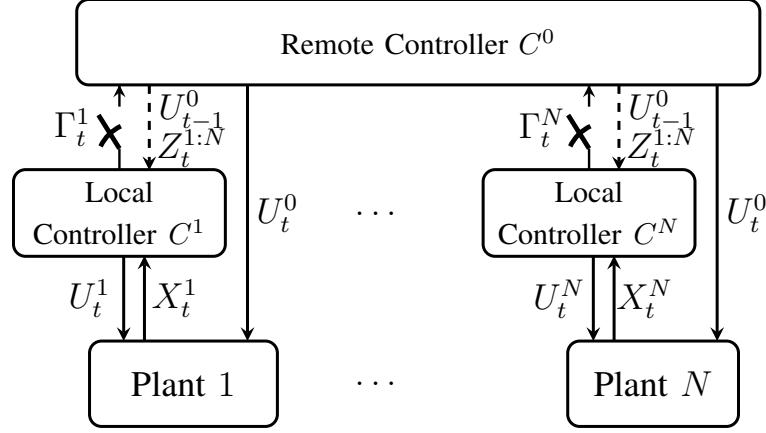


Fig. 1. System model. The binary random variables  $\Gamma_t^{1:N}$  indicates whether packets are transmitted successfully.

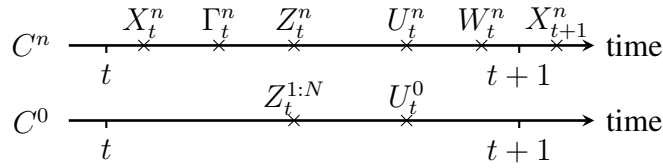


Fig. 2. The time-ordering of relevant variables.

$X_0^n$  is a random vector with distribution  $\pi_{X_0^n}$ ,  $W_t^n \in \mathbb{R}^{d_x}$  is a zero-mean noise vector at time  $t$  with distribution  $\pi_{W_t^n}$ .  $X_0^{1:N}$ ,  $W_{0:T}^{1:N}$  are independent random vectors with finite second moments. Note that we do not assume that  $X_0^{1:N}$  and  $W_{0:T}^{1:N}$  are Gaussian.

The overall dynamics can be written as

$$X_{t+1} = AX_t + BU_t + W_t \quad (2)$$

where  $X_t = \text{vec}(X_t^{1:N})$ ,  $U_t = \text{vec}(U_t^{0:N})$ ,  $W_t = \text{vec}(W_t^{0:N})$  and  $A, B$  are defined as

$$A = \begin{bmatrix} A^{11} & \mathbf{0} \\ & \ddots \\ \mathbf{0} & A^{NN} \end{bmatrix}, B = \begin{bmatrix} B^{10} & B^{11} & \mathbf{0} \\ \vdots & & \ddots \\ B^{N0} & \mathbf{0} & B^{NN} \end{bmatrix}. \quad (3)$$

At each time  $t$  the local controller  $C^n$ ,  $n \in \mathcal{N}$ , perfectly observes the state  $X_t^n$  and sends the observed state to the remote controller  $C^0$  through an unreliable channel with link failure

probability  $p^n$ . Let  $\Gamma_t^n$  be Bernoulli random variable describing the state of this channel, that is,  $\Gamma_t^n = 0$  when the link is broken and  $\Gamma_t^n = 1$  otherwise. We assume that  $\Gamma_{0:T}^{1:N}$  are independent and identically distributed (i.i.d.) random variables and they are independent of  $X_0^{1:N}$  and  $W_{0:T}^{1:N}$ . Furthermore, let  $Z_t^n$  be the output of the channel between the local controller  $C^n$  and the remote controller  $C^0$ . Then,

$$\Gamma_t^n = \begin{cases} 1 & \text{with probability } (1 - p^n), \\ 0 & \text{with probability } p^n. \end{cases} \quad (4)$$

$$Z_t^n = \begin{cases} X_t^n & \text{when } \Gamma_t^n = 1, \\ \emptyset & \text{when } \Gamma_t^n = 0. \end{cases} \quad (5)$$

We assume that the channel outputs  $Z_t^{1:N}$  are perfectly observed by  $C^0$ . Furthermore, we assume that there exists a perfect link from  $C^0$  to  $C^n$ , for  $n \in \mathcal{N}$ . Therefore,  $C^0$  can share  $Z_t^{1:N}$  and  $U_{t-1}^0$  with  $C^{1:N}$ . All controllers select their control actions after observing  $Z_t^{1:N}$ . A schematic of the time ordering of the variables is shown in Fig. 2. We assume that for all  $n \in \mathcal{N}$ , the links from  $C^n$  and  $C^0$  to the plant  $n$  are perfect.

Let  $H_t^n$  denote the information available to controller  $C^n$ ,  $n \in \bar{\mathcal{N}}$ , to make decisions at time  $t$ . Then,

$$\begin{aligned} H_t^n &= \{X_{0:t}^n, U_{0:t-1}^n, Z_{0:t}^{1:N}, U_{0:t-1}^0\}, \quad \forall n \in \mathcal{N} \\ H_t^0 &= \{Z_{0:t}^{1:N}, U_{0:t-1}^0\}. \end{aligned} \quad (6)$$

Let  $\mathcal{H}_t^n$  be the space of all realizations of  $H_t^n$ . Then,  $C^n$ 's actions are selected according to

$$U_t^n = g_t^n(H_t^n), \quad \forall n \in \bar{\mathcal{N}}, \quad (7)$$

where  $g_t^n : \mathcal{H}_t^n \mapsto \mathbb{R}^{d_U^n}$  is a Borel measurable mapping. The collection of mappings  $g_0^n, \dots, g_T^n$  is called the strategy of the controller  $C^n$  as is denoted by  $g^n$ . The collection of all controllers' strategies  $g^{0:N}$  is called the strategy profile.

The instantaneous cost  $c_t(X_t^{1:N}, U_t^{0:N})$  of the system is a general quadratic function given by

$$\begin{aligned} c_t(X_t^{1:N}, U_t^{0:N}) &= S_t^T R_t S_t, \quad \text{where} \\ S_t &= \text{vec}(X_t^{1:N}, U_t^{0:N}), \quad R_t = \begin{bmatrix} R_t^{XX} & R_t^{XU} \\ R_t^{UX} & R_t^{UU} \end{bmatrix}, \end{aligned} \quad (8)$$

and

$$R_t^{XX} = \begin{bmatrix} R_t^{X^1 X^1} & \dots & R_t^{X^1 X^N} \\ \vdots & \ddots & \vdots \\ R_t^{X^N X^1} & \dots & R_t^{X^N X^N} \end{bmatrix} =: [R_t^{X^i X^j}]_{i,j \in \mathcal{N}},$$

$$R_t^{XU} = (R_t^{UX})^\top = [R_t^{X^i U^j}]_{i \in \mathcal{N}, j \in \bar{\mathcal{N}}}, R_t^{UU} = [R_t^{U^i U^j}]_{i,j \in \bar{\mathcal{N}}}. \quad (9)$$

$R_t$  is a symmetric positive semi-definite (PSD) matrix and  $R_t^{UU}$  is a symmetric positive definite (PD) matrix.

The performance of strategies  $g^n := g_{0:T}^n$ ,  $n \in \bar{\mathcal{N}}$  is the total expected cost given by

$$J(g^{0:N}) = \mathbb{E}^{g^{0:N}} \left[ \sum_{t=0}^T c_t(X_t^{1:N}, U_t^{0:N}) \right]. \quad (10)$$

Let  $\mathcal{G}^n$  be the set of all control strategies for  $C^n$ ,  $n \in \bar{\mathcal{N}}$ . Then, the optimal control problem for  $C^{0:N}$  is formally defined below.

**Problem 1.** *For the system model described by (1)-(10), we would like to solve the following strategy optimization problem,*

$$\inf_{g^n \in \mathcal{G}^n, n \in \bar{\mathcal{N}}} J(g^{0:N}). \quad (11)$$

**Remark 1.** *Without loss of optimality, we can restrict attention to strategy profiles  $g^{0:N}$  that ensure a finite expected cost at each time step. Because  $R_t$  is positive semi-definite and  $R_t^{UU}$  is positive definite, finite expected cost at all time  $t$  is equivalent to*

$$\mathbb{E}^{g^{0:N}} [(U_t^n)^\top U_t^n] = \mathbb{E}^{g^{0:N}} [g_t^n (H_t^n)^\top g_t^n (H_t^n)] < \infty, \quad \forall n \in \bar{\mathcal{N}}, \forall t. \quad (12)$$

*Therefore, in the subsequent analysis we will implicitly assume that the strategy profile under consideration,  $g^{0:N}$ , ensures that for all time  $t$  and for all  $n \in \bar{\mathcal{N}}$ ,  $g_t^n : \mathcal{H}_t^n \mapsto \mathbb{R}^{d_v^n}$  has finite second moments, that is, (12) holds.*

Problem 1 is a  $(N+1)$ -controller decentralized optimal control problem. Decentralized optimal control problems are generally believed to be hard. For decentralized linear-quadratic-Gaussian (LQG) control problems with partially-nested information structure, linear control strategies are optimal [30]. An information structure is partially-nested if whenever the action of a controller affects the information of another controller, the latter knows whatever the former knows. Note

that Problem 1 is not a partially nested problem. In particular,  $C^n$ 's action  $U_{t-1}^n$ ,  $n \in \mathcal{N}$ , affects  $X_t^n$ , and consequently, it affects  $Z_t^n$ . Since  $Z_t^n$  is a part of the remote controller  $C^0$ 's information  $H_t^0$  at  $t$  but  $H_{t-1}^n \not\subset H_t^0$ , the information structure in Problem 1 is not partially nested. Furthermore, in Problem 1,  $X_0^{1:N}$  and  $W_{0:T}^{1:N}$  are not necessarily Gaussian. Therefore, linear control strategies are not necessarily optimal for Problem 1.

Our approach to Problem 1 is based on the common information approach [40] for decentralized decision-making. We identify the common information among the  $N + 1$  controllers and use it to define a common belief on the system state. This common belief can serve as an information state for a dynamic program that characterizes optimal control strategies.

### III. EQUIVALENT PROBLEM AND DYNAMIC PROGRAM

We first provide a structural result for the local controllers' strategies.

**Lemma 1.** *Let  $\hat{H}_t^n = \{X_t^n, H_t^0\}$ , and  $\hat{\mathcal{G}}^n = \{g^n \in \mathcal{G}^n : g^n \text{ depends only on } \hat{H}_t^n\}$ . Then,*

$$\inf_{g^n \in \mathcal{G}^n, n \in \mathcal{N}} J(g^{0:N}) = \inf_{g^n \in \hat{\mathcal{G}}^n, n \in \mathcal{N}, g^0 \in \mathcal{G}^0} J(g^{0:N}). \quad (13)$$

*Proof:* See Appendix B for a proof. ■

Due to Lemma 1, we only need to consider strategies  $g^n \in \hat{\mathcal{G}}^n$  for the local controller  $C^n$ ,  $n \in \mathcal{N}$ . That is, the local controller  $C^n$  only needs to use  $\hat{H}_t^n = \{X_t^n, H_t^0\}$  to make the decision at  $t$ .

According to the information structure (6) and Lemma 1,  $H_t^0$  is the common information among  $C^{0:N}$ , and  $X_t^n = \hat{H}_t^n \setminus H_t^0$  is the *private* information used by the local controller  $C^n$  in its decision-making. Note that  $C^0$  has no private information, since  $H_t^0 \setminus H_t^0 = \emptyset$ . Based on the common information approach [40], we construct below an equivalent centralized problem using the controllers' common information.

#### A. Equivalent Centralized Problem

Consider arbitrary control strategies  $g^n \in \hat{\mathcal{G}}^n$ ,  $n \in \mathcal{N}$  and  $g^0 \in \mathcal{G}^0$  for the local and the remote controllers, respectively. Under these strategies

$$U_t^n = g_t^n(X_t^n, H_t^0) = \mathbb{E}^g[g_t^n(X_t^n, H_t^0)|H_t^0] + \left\{ g_t^n(X_t^n, H_t^0) - \mathbb{E}^g[g_t^n(X_t^n, H_t^0)|H_t^0] \right\}. \quad (14)$$



We can rewrite (14) as

$$U_t^n = \bar{g}_t^n(H_t^0) + \tilde{g}_t^n(X_t^n, H_t^0) \quad (15)$$

where

$$\begin{aligned} \bar{g}_t^n(H_t^0) &= \mathbb{E}^g[g_t^n(X_t^n, H_t^0)|H_t^0], \\ \tilde{g}_t^n(X_t^n, H_t^0) &= g_t^n(X_t^n, H_t^0) - \mathbb{E}^g[g_t^n(X_t^n, H_t^0)|H_t^0]. \end{aligned} \quad (16)$$

Observe that  $\tilde{g}_t^n(X_t^n, H_t^0)$  is conditionally zero-mean given  $H_t^0$ , that is,  $\mathbb{E}^g[\tilde{g}_t^n(X_t^n, H_t^0)|H_t^0] = 0$ .

Note that  $\bar{g}_t^n(H_t^0)$  is the conditional mean of  $g_t^n(X_t^n, H_t^0)$  given the remote controller's information  $H_t^0$  and  $\tilde{g}_t^n(X_t^n, H_t^0)$  can be interpreted as the deviation of  $g_t^n(X_t^n, H_t^0)$  from the mean  $\bar{g}_t^n(H_t^0)$ . Considering this representation, (15) suggests that at each time  $t$ , the problem of finding optimal control action  $U_t^n$  for  $C^n$  is equivalent to the problem of finding “mean value” of  $U_t^n$  and “deviation” of  $U_t^n$  from the mean value.

We will use the above representation of  $g_t^n$  in terms of  $\bar{g}_t^n$  and  $\tilde{g}_t^n$  to formulate a centralized decision-making problem. In the centralized problem, the remote controller is the only decision-maker. At each time  $t$ , given the realization  $h_t^0$  of the remote controller's information, it makes three decisions:

- 1) Remote controller's control action  $u_t^0 = \phi_t^0(h_t^0)$ ,
- 2) Mean value of all local controller's control action  $\bar{u}_t^n = \bar{\phi}_t^n(h_t^0)$ ,  $n \in \mathcal{N}$ ,
- 3) A “deviation from the mean value” mapping  $q_t^n \in \mathcal{Q}^n$ ,  $n \in \mathcal{N}$ , where  $\mathcal{Q}^n = \{q^n : \mathbb{R}^{d_x^n} \mapsto \mathbb{R}^{d_u^n}, \text{ Borel measurable}\}$  and  $q_t^n = \tilde{\phi}_t^n(h_t^0)$ .

The control actions applied to the system described by (1)-(5) are:

- $u_t^0$  as the control action of the remote controller,
- $u_t^n = \bar{u}_t^n + q_t^n(X_t^n)$  as the control action of the  $n$ -th local controller,  $n \in \mathcal{N}$ .

We call  $u_t^{prs} = (u_t^0, \bar{u}_t^{1:N}, q_t^{1:N})$  the prescription at time  $t$ . We denote  $(\phi_t^0, \bar{\phi}_t^{1:N}, \tilde{\phi}_t^{1:N})$  by  $\phi_t^{prs}$  and write  $u_t^{prs} = \phi_t^{prs}(h_t^0)$  to indicate that the prescription is a function of the common information  $h_t^0$ . The functions  $(\phi_t^{prs}, 0 \leq t \leq T)$  are collectively referred to as the prescription strategy and denoted by  $\phi^{prs}$ . The prescription strategy is required to satisfy the following conditions:

(C1)  $\phi^0 \in \mathcal{G}^0$ .

(C2) Define  $\phi_t^n(X_t^n, H_t^0) := \bar{\phi}_t^n(H_t^0) + [\tilde{\phi}_t^n(H_t^0)](X_t^n)$ . Then,  $\phi^n \in \hat{\mathcal{G}}^n$  for any  $n \in \mathcal{N}$ .

(C3) We require that for any  $t$ ,

$$\mathbb{E}^{\phi^{prs}}[\tilde{\phi}_t^n(H_t^0)(X_t^n)|H_t^0] = 0, \quad (17)$$

where  $\mathbb{E}^{\phi^{prs}}$  is the probability measure induced by the prescription strategy  $\phi^{prs}$ .

Denote by  $\Phi^{prs}$  the set of all prescription strategies satisfying the above conditions. Consider the following problem of optimizing the prescription strategies.

**Problem 2.** Consider the system described by (1)-(9). Given a prescription strategy  $\phi^{prs} \in \Phi^{prs}$ , let

$$\Lambda(\phi^{prs}) = \mathbb{E}^{\phi^{prs}} \left[ \sum_{t=0}^T c_t^{prs}(X_t^{1:N}, U_t^{prs}) \right] \quad (18)$$

where for any  $x_t^{1:N}$  and  $u_t^{prs} = (u_t^0, \bar{u}_t^{1:N}, q_t^{1:N})$ ,

$$c_t^{prs}(x_t^{1:N}, u_t^{prs}) = c_t(x_t^{1:N}, u_t^0, \{\bar{u}_t^n + q_t^n(x_t^n)\}_{n \in \mathcal{N}}) \quad (19)$$

Then, we would like to solve the following optimization problem,

$$\inf_{\phi^{prs} \in \Phi^{prs}} \Lambda(\phi^{prs}). \quad (20)$$

Note that any feasible prescription strategy in Problem 2 results in control strategies in Problem 1. On the other hand, any control strategies in Problem 1 can be represented by a prescription strategy in Problem 2. This equivalence between Problems 1 and 2 is formally stated in the following lemma.

**Lemma 2.** Problems 1 and 2 are equivalent in the following sense:

- 1) For any control strategies  $g^n \in \hat{\mathcal{G}}^n$  and  $g^0 \in \mathcal{G}^0$  in Problem 1, there is a prescription strategy  $\phi^{prs} \in \Phi^{prs}$  in Problem 2 such that for  $0 \leq t \leq T$ ,

$$\phi_t^0(H_t^0) = g_t^0(H_t^0), \quad (21)$$

$$\bar{\phi}_t^n(H_t^0) = \bar{g}_t^n(H_t^0) = \mathbb{E}^g[g_t^n(X_t^n, H_t^0)|H_t^0], \quad \forall n \in \mathcal{N}, \quad (22)$$

$$[\tilde{\phi}_t^n(H_t^0)](X_t^n) = \tilde{g}_t^n(X_t^n, H_t^0) = g_t^n(X_t^n, H_t^0) - \mathbb{E}^g[g_t^n(X_t^n, H_t^0)|H_t^0], \quad \forall n \in \mathcal{N}, \quad (23)$$

$$\Lambda(\phi^{prs}) = J(g^{0:N}). \quad (24)$$

2) Conversely, for any prescription strategy  $\phi^{prs} \in \Phi^{prs}$  in Problem 2, there are control strategies  $g^n \in \hat{\mathcal{G}}^n$  and  $g^0 \in \mathcal{G}^0$  in Problem 1 such that for  $0 \leq t \leq T$ ,

$$g_t^0(H_t^0) = \phi_t^0(H_t^0), \quad (25)$$

$$g_t^n(X_t^n, H_t^0) = \bar{g}_t^n(H_t^0) + \tilde{g}_t^n(X_t^n, H_t^0) = \bar{\phi}_t^n(H_t^0) + [\tilde{\phi}_t^n(H_t^0)](X_t^n), \quad \forall n \in \mathcal{N}, \quad (26)$$

$$J(g^{0:N}) = \Lambda(\phi^{prs}). \quad (27)$$

*Proof:* See Appendix C for a proof. ■

### B. Information State for Problem 2

Since Problem 2 is a centralized decision-making problem for the remote controller  $C^0$ ,  $C^0$ 's belief on the system states can be used as an information state for decision-making. Note that  $C^0$ 's information at any time  $t$  is the *common information*  $H_t^0$ . Therefore, we define the common belief  $\Theta_t$  as the conditional probability distribution of  $X_t^{1:N}$  given  $H_t^0$ . That is, under prescription strategies  $\phi_{0:t-1}^{prs} \in \Phi^{prs}$  until time  $t-1$ , for any measurable set  $E \subset \prod_{n=1}^N \mathbb{R}^{d_x^n}$ ,

$$\Theta_t(E) := \mathbb{P}^{\phi_{0:t-1}^{prs}}(\text{vec}(X_t^{1:N}) \in E | H_t^0). \quad (28)$$

Let  $\Theta_t^n$  denote the marginal common belief on  $X_t^n$ . That is, for any measurable set  $E^n \subset \mathbb{R}^{d_x^n}$

$$\Theta_t^n(E^n) := \mathbb{P}^{\phi_{0:t-1}^{prs}}(X_t^n \in E^n | H_t^0). \quad (29)$$

Then, for a given realization  $h_t^0$  of  $H_t^0$ , the corresponding realization  $\theta_t$  of  $\Theta_t$  belongs to  $\Delta(\prod_{n=1}^N \mathbb{R}^{d_x^n})$  and the realization  $\theta_t^n$  of  $\Theta_t^n$  belongs to  $\Delta(\mathbb{R}^{d_x^n})$ ,  $n \in \mathcal{N}$ .

Since the plants' dynamics are only coupled through the remote controller's actions which belongs to the common information, the common belief has the following conditional independence property.

**Lemma 3.** Consider a feasible prescription strategy  $\phi^{prs} \in \Phi^{prs}$ . Then, the random vectors  $X_t^{1:N}$  are conditionally independent given the common information  $H_t^0$ . That is, for any measurable sets  $E^n \subset \mathbb{R}^{d_x^n}$ ,  $n \in \mathcal{N}$ ,

$$\Theta_t(\prod_{n=1}^N E^n) = \prod_{n=1}^N \Theta_t^n(E^n) \quad (30)$$

where  $\Theta_t$  and  $\Theta_t^n$  are given by (28) and (29).

*Proof:* The proof is a direct consequence of Part 2 of Claim 2 in Appendix A. ■

From Lemma 3, the joint common belief  $\Theta_t$  can be represented by the collection of marginal common beliefs  $\Theta_t^{1:N}$ .

We show in the following that the marginal common beliefs  $\theta_t^n, n \in \mathcal{N}$ , can be sequentially updated.

**Lemma 4.** *For any feasible prescription strategy  $\phi^{prs} \in \Phi^{prs}$  and for any  $h_t^0 \in \mathcal{H}_t^0$ , we recursively define  $\nu_t^n(h_t^0) \in \Delta(\mathbb{R}^{d_X^n})$  as follows:*

*For any measurable set  $E^n \subset \mathbb{R}^{d_X^n}$ ,*

$$[\nu_0^n(h_0^0)](E^n) = \begin{cases} \pi_{X_0^n}(E^n) & \text{if } z_0^n = \emptyset, \\ \mathbb{1}_{E^n}(x_0^n) & \text{if } z_0^n = x_0^n. \end{cases} \quad (31)$$

$$[\nu_{t+1}^n(h_{t+1}^0)](E^n) = [\psi_t^n(\nu_t^n(h_t^0), u_t^{prs}, z_{t+1}^n)](E^n), \quad (32)$$

where  $u_t^{prs} = \phi_t^{prs}(h_t^0)$  and  $\psi_t^n(\nu_t^n(h_t^0), u_t^{prs}, z_{t+1}^n)$  is defined as follows:

- If  $z_{t+1}^n = x_{t+1}^n$ , then

$$[\psi_t^n(\nu_t^n(h_t^0), u_t^{prs}, z_{t+1}^n)](E^n) = \mathbb{1}_{E^n}(x_{t+1}^n). \quad (33)$$

- If  $z_{t+1}^n = \emptyset$ , then

$$[\psi_t^n(\nu_t^n(h_t^0), u_t^{prs}, \emptyset)](E^n) = \int \int \mathbb{1}_{E^n}(f_t^n(x_t^n, w_t^n, u_t^{prs})) \nu_t^n(h_t^0)(dx_t^n) \pi_{W_t^n}(dw_t^n), \quad (34)$$

where

$$f_t^n(x_t^n, w_t^n, u_t^{prs}) = A^{nn}x_t^n + B^{nn}(\bar{u}_t^n + q_t^n(x_t^n)) + B^{n0}u_t^0 + w_t^n. \quad (35)$$

Then,  $\nu_t^n$  is a conditional probability of  $X_t^n$  given  $H_t^0$ , that is  $[\nu_t^n(H_t^0)](E^n) = \mathbb{P}^{\phi_{0:t-1}^{prs}}(X_t^n \in E^n | H_t^0)$ .

*Proof:* See Appendix D for a proof. ■

Lemma 4 implies that the realization  $\theta_t^n$  of the belief  $\Theta_t^n$  can be updated according to

$$\theta_{t+1}^n = \psi_t^n(\theta_t^n, u_t^{prs}, z_{t+1}^n). \quad (36)$$

Recall that  $\mathcal{Q}^n$  is the space of all measurable functions  $q : \mathbb{R}^{d_X^n} \mapsto \mathbb{R}^{d_U^n}$ . We now define the space of mapping  $\mathcal{Q}^n(\theta^n) \subset \mathcal{Q}^n$  for any  $\theta^n \in \Delta(\mathbb{R}^{d_X^n})$  to be

$$\mathcal{Q}^n(\theta^n) = \left\{ q^n : \mathbb{R}^{d_x^n} \mapsto \mathbb{R}^{d_u^n} \text{ measurable, } \int q^n(x^n) \theta^n(dx^n) = 0 \right\}. \quad (37)$$

Note that for any feasible prescription strategy  $\phi^{prs} \in \Phi^{prs}$ , (17) implies that for almost every realization  $h_t^0$  under  $\phi^{prs}$ ,

$$\mathbb{E}^{\phi^{prs}}[q_t^n(X_t^n)|h_t^0] = 0, \quad (38)$$

where  $q_t^n = \tilde{\phi}_t^n(h_t^0)$ . Then, (38) and (29) imply that for almost every realization  $h_t^0$ ,  $\int q_t^n(x_t^n) \theta_t^n(dx_t^n) = 0$ , that is,  $q_t^n$  belongs to  $\mathcal{Q}^n(\theta_t^n)$ .

### C. Dynamic Program for Problem 2

We can use the collection of marginal common beliefs  $\Theta_t^{1:N}$  as an information state to construct a dynamic program for Problem 2. For that purpose, we will use the following definitions.

For every  $x \in \mathbb{R}^{d_x}$ , we use  $\rho(x)$  to denote the Dirac-delta distribution at  $x$ . Then, for any  $E \subset \mathbb{R}^{d_x}$ ,  $[\rho(x)](E) = \mathbb{1}_E(x)$ .

For any  $\theta_t^n \in \Delta(\mathbb{R}^{d_x^n})$ ,  $q_t^n \in \mathcal{Q}^n(\theta_t^n)$ ,  $\bar{u}_t^n \in \mathbb{R}^{d_u^n}$  for  $n \in \mathcal{N}$  and  $u_t^0 \in \mathbb{R}^{d_u^0}$ , letting  $u_t^{prs} = (u_t^0, \bar{u}_t^{1:N}, q_t^{1:N})$ , we define

- $IC(\theta_t^{1:N}, u_t^{prs}) := \int c_t^{prs}(x_t^{1:N}, u_t^{prs}) \prod_{n \in \mathcal{N}} \theta_t^n(dx_t^n)$ . This function represents the remote controller's expected instantaneous cost at time  $t$  when its beliefs on the system states are  $\theta_t^{1:N}$  and it selects  $u_t^{prs}$ .
- $\alpha_t^n := \psi_t^n(\theta_t^n, u_t^{prs}, \emptyset)$  (see (36) and note that  $\alpha_t^n \in \Delta(\mathbb{R}^{d_x^n})$ ).
- For any realization  $\gamma_{t+1}^n \in \{0, 1\}$  of  $\Gamma_{t+1}^n$ ,  $NB(\gamma_{t+1}^n, \alpha_t^n, x_{t+1}^n) := (1 - \gamma_{t+1}^n) \alpha_t^n + \gamma_{t+1}^n \rho(x_{t+1}^n)$ . This function represents the next belief equation for  $\theta_t^n$ . If  $\gamma_{t+1}^n = 0$ ,  $\theta_{t+1}^n = \alpha_t^n$  and if  $\gamma_{t+1}^n = 1$ ,  $\theta_{t+1}^n = \rho(x_{t+1}^n)$ .
- $LS(p^n, \gamma_{t+1}^n) := (p^n)^{1-\gamma_{t+1}^n} (1 - p^n)^{\gamma_{t+1}^n}$ . If  $\gamma_{t+1}^n = 0$ , this function represents the link failure probability, that is  $p^n$ . If  $\gamma_{t+1}^n = 1$ , this function represents the probability that link is active, that is  $1 - p^n$ .

The following theorem provides a dynamic program for optimal prescription strategies of Problem 2.

**Theorem 1.** Suppose there exist functions  $\{V_t : \prod_{m=1}^N \Delta(\mathbb{R}^{d_x^m}) \mapsto \mathbb{R} \text{ for } t = 0, 1, \dots, T+1\}$  such that for each  $\theta_t^{1:N} \in \prod_{m=1}^N \Delta(\mathbb{R}^{d_x^m})$ , the following are true:

- $V_{T+1}(\theta_t^{1:N}) = 0$ ,
- For any  $t = 0, 1, \dots, T$

$$V_t(\theta_t^{1:N}) = \min_{\{q_t^n \in \mathcal{Q}^n(\theta_t^n)\}_{n \in \mathcal{N}}} \left\{ \min_{\{\bar{u}_t^n \in \mathbb{R}^{d_u^n}\}_{n \in \mathcal{N}}, u_t^0 \in \mathbb{R}^{d_u^0}} \left\{ IC(\theta_t^{1:N}, u_t^{prs}) \right. \right. \\ \left. \left. + \sum_{\gamma_{t+1}^1 \in \{0,1\}} \dots \sum_{\gamma_{t+1}^N \in \{0,1\}} \prod_{n \in \mathcal{N}} LS(p^n, \gamma_{t+1}^n) \int V_{t+1} \left( \left\{ NB(\gamma_{t+1}^n, \alpha_t^n, x_{t+1}^n) \right\}_{n \in \mathcal{N}} \right) \prod_{n \in \mathcal{N}} \alpha_t^n(dx_{t+1}^n) \right\} \right\}, \quad (39)$$

where  $u_t^{prs} = (u_t^0, \bar{u}_t^{1:N}, q_t^{1:N})$ .

Further, suppose there exists a feasible prescription strategy  $\phi^{prs*} \in \Phi^{prs}$  such that for any realization  $h_t^0 \in \mathcal{H}_t^0$  and its corresponding common belief  $\theta_t^n = \nu_t^n(h_t^0)$ ,  $n \in \mathcal{N}$  (defined by Lemma 4), the prescription  $u_t^{prs*} = (u_t^{0*}, \bar{u}_t^{1:N*}, q_t^{1:N*}) = \phi^{prs*}(h_t^0)$  achieves the minimum in the definition of  $V_t(\theta_t^{1:N})$ . Then,  $\phi^{prs*}$  is an optimal prescription strategy for Problem 2.

*Proof:* See Appendix E for a proof. ■

If the functions  $V_{0:T}$  of Theorem 1 can be shown to exist, then Theorem 1 provides a dynamic program to solve Problem 2. Even if such a dynamic program is available, it suffers from two significant challenges. First, it is a dynamic program on the belief space  $\prod_{n=1}^N \Delta(\mathbb{R}^{d_x^n})$  which is infinite dimensional. Second, each step of the dynamic program involves a functional optimization over the space  $\mathcal{Q}^n(\theta_t^n)$ ,  $n \in \mathcal{N}$ . In the next section, we show that the functions satisfying (39) exist and it is possible to use the dynamic program of Theorem 1 to obtain optimal control strategies in Problem 1.

#### IV. OPTIMAL CONTROL STRATEGIES

##### A. Optimal Prescription strategy in Problem 2

For a vector  $x$  and a matrix  $G$ , we use

$$QF(G, x) = x^\top G x = \text{tr}(Gxx^\top) \quad (40)$$

to denote the quadratic form.

The following theorem presents the structure of the functions  $V_{0:T}$  and an explicit optimal solution of the dynamic program in Theorem 1.

**Theorem 2.** For  $t = 0, 1, \dots, T$ , the functions  $V_t(\cdot)$  of Theorem 1 exist and are given by<sup>1</sup>

$$V_t(\theta_t^{1:N}) = QF\left(P_t, \text{vec}\left(\{\mu(\theta_t^n)\}_{n \in \mathcal{N}}\right)\right) + \sum_{n=1}^N \text{tr}\left(\tilde{P}_t^{nn} \text{cov}(\theta_t^n)\right) + e_t, \quad (41)$$

where

$$e_t = \sum_{s=t}^T \sum_{n=1}^N \text{tr}\left(\left((1-p^n)P_{s+1}^{nn} + p^n \tilde{P}_{s+1}^{nn}\right) \text{cov}(\pi_{W_s^n})\right). \quad (42)$$

The matrices  $P_t, G_t, L_t, \tilde{P}_t, \tilde{G}_t, \tilde{L}_t^{nn}, \hat{L}_t^{nn}$ ,  $n \in \mathcal{N}$ , defined recursively below are symmetric positive semi-definite (PSD);  $G_t^{UU}$  and  $\tilde{G}_t^{UU}$  are positive definite (PD).

$$P_{T+1} = [P_{T+1}^{ij}]_{i,j \in \mathcal{N}} = \mathbf{0}, \quad (43)$$

$$L_t = \begin{bmatrix} A & B \end{bmatrix}^\top P_{t+1} \begin{bmatrix} A & B \end{bmatrix}, \quad (44)$$

$$G_t = \begin{bmatrix} G_t^{XX} & G_t^{XU} \\ G_t^{UX} & G_t^{UU} \end{bmatrix} = R_t + L_t, \quad (45)$$

$$P_t = [P_t^{ij}]_{i,j \in \mathcal{N}} = G_t^{XX} - G_t^{XU} (G_t^{UU})^{-1} G_t^{UX}, \quad (46)$$

$$\tilde{P}_{T+1}^{nn} = \mathbf{0}, \quad \forall n \in \mathcal{N}, \quad (47)$$

$$\hat{L}_t^{nn} = \left( \begin{bmatrix} A & B \end{bmatrix}_{r,n} \right)^\top P_{t+1}^{nn} \begin{bmatrix} A & B \end{bmatrix}_{r,n}, \quad (48)$$

$$\tilde{L}_t^{nn} = \left( \begin{bmatrix} A & B \end{bmatrix}_{r,n} \right)^\top \tilde{P}_{t+1}^{nn} \begin{bmatrix} A & B \end{bmatrix}_{r,n}, \quad (49)$$

$$\tilde{G}_t = \begin{bmatrix} \tilde{G}_t^{XX} & \tilde{G}_t^{XU} \\ \tilde{G}_t^{UX} & \tilde{G}_t^{UU} \end{bmatrix} = R_t + \sum_{n \in \mathcal{N}} \left( (1-p^n) \hat{L}_t^{nn} + p^n \tilde{L}_t^{nn} \right), \quad (50)$$

$$\tilde{P}_t^{nn} = \tilde{G}_t^{X^n X^n} - \tilde{G}_t^{X^n U^n} (\tilde{G}_t^{U^n U^n})^{-1} \tilde{G}_t^{U^n X^n}, \quad \forall n \in \mathcal{N}. \quad (51)$$

Furthermore, the optimal prescription strategy is given as,

$$\begin{bmatrix} u_t^{0*} \\ \bar{u}_t^{1*} \\ \vdots \\ \bar{u}_t^{N*} \end{bmatrix} = \begin{bmatrix} \phi_t^{0*}(\theta_t^{1:N}) \\ \bar{\phi}_t^{1*}(\theta_t^{1:N}) \\ \vdots \\ \bar{\phi}_t^{N*}(\theta_t^{1:N}) \end{bmatrix} = -(G_t^{UU})^{-1} G_t^{UX} \begin{bmatrix} \mu(\theta_t^1) \\ \vdots \\ \mu(\theta_t^N) \end{bmatrix}, \quad (52)$$

$$q_t^{n*}(x_t^n) = [\tilde{\phi}_t^{n*}(\theta_t^{1:N})](x_t^n) = -(\tilde{G}_t^{U^n U^n})^{-1} \tilde{G}_t^{U^n X^n} (x_t^n - \mu(\theta_t^n)). \quad (53)$$

*Proof:* See Appendix F for a proof. ■

<sup>1</sup>Recall that  $\mu(\theta_t^n)$  and  $\text{cov}(\theta_t^n)$  are the mean vector and covariance matrix for the distribution  $\theta_t^n$ .

### B. Optimal control strategies in Problem 1

From Theorem 1 and Theorem 2, we can now explicitly compute the optimal control strategies for Problem 1.

**Theorem 3.** *The optimal strategies of Problem 1 are given by*

$$\begin{bmatrix} U_t^{0*} \\ \bar{U}_t^{1*} \\ \vdots \\ \bar{U}_t^{N*} \end{bmatrix} = -(G_t^{UU})^{-1} G_t^{UX} \begin{bmatrix} \hat{X}_t^1 \\ \vdots \\ \hat{X}_t^N \end{bmatrix}, \quad (54)$$

$$U_t^{n*} = \bar{U}_t^{n*} - \left( \tilde{G}_t^{U^n U^n} \right)^{-1} \tilde{G}_t^{U^n X^n} \left( X_t^n - \hat{X}_t^n \right), \quad n \in \mathcal{N} \quad (55)$$

where  $\hat{X}_t^n$  is the estimate (conditional expectation) of  $X_t^n$  based on the common information  $H_t^0$ .  $\hat{X}_t^n$  can be computed recursively according to

$$\hat{X}_0^n = \begin{cases} \mu(\pi_{X_0^n}) & \text{if } Z_0^n = \emptyset, \\ X_0^n & \text{if } Z_0^n = X_0^n. \end{cases} \quad (56)$$

$$\hat{X}_{t+1}^n = \begin{cases} A^{nn} \hat{X}_t^n + B^{nn} \bar{U}_t^{n*} + B^{n0} U_t^{0*} & \text{if } Z_{t+1}^n = \emptyset, \\ X_{t+1}^n & \text{if } Z_{t+1}^n = X_{t+1}^n. \end{cases} \quad (57)$$

*Proof:* See Appendix G for a proof. ■

Theorem 3 shows that the optimal control strategy of the remote controller  $C^0$  is linear in the estimated state  $\hat{X}_t^{1:N}$ , and the optimal control strategy of the local controller  $C^n$ ,  $n \in \mathcal{N}$ , is linear in both the state  $X_t^n$  and the estimated state  $\hat{X}_t^{1:N}$ .

### C. Simplification of communication

The assumption that the remote controller can perfectly send  $Z_t^{1:N}$  and  $U_{t-1}^0$  to each local controller requires a lot of communication resources such as bandwidth, especially as the transmitted information grows with the number  $N$  of local controllers. One method to simplify the communication is to transmit only the information needed for computing the optimal control actions. According to Theorem 3, the local controller  $C^n$  only needs to know  $\bar{U}_t^{n*}$  and  $\hat{X}_t^n$  (in addition to  $X_t^n$ ) to compute the optimal control strategy at time  $t$ . Hence, instead of sending  $Z_t^{1:N}$  and  $U_{t-1}^0$  at time  $t$ , the remote controller can send only  $\bar{U}_t^{n*}$  and  $\hat{X}_t^n$  to each local controller



$C^n$ . By doing so, the communication resources required for each transmission are fixed and do not change with the number of controllers.

The above observation also shows that the resource-constrained channels can achieve the same performance as perfect channels as long as the channel allows the vector  $\text{vec}(\hat{X}_t^n, \bar{U}_t^{n*}) \in \mathbb{R}^{d_X^n + d_U^n}$  to be perfectly transmitted from  $C^0$  to  $C^n$ .

Note that the communication can be further simplified when  $d_U^0 < d_X^n$ . In this case, the remote controller can send  $\Gamma_t^n$  and  $\text{vec}(U_{t-1}^0, \bar{U}_t^{n*}) \in \mathbb{R}^{d_U^0 + d_L^n}$  to the local controller  $C^n$ . Then, the local controller  $C^n$  can first compute  $\hat{X}_t^n$  using (57) and then, uses  $\hat{X}_t^n$  and  $\bar{U}_t^{n*}$  to compute the optimal control action  $U_t^{n*}$  based on (55).

#### D. Special cases

1) *No control action for some controllers:* Our model can also capture the situation when some controllers share their observations but do not take any action. In particular, the situation when controller  $C^n$ ,  $n \in \bar{\mathcal{N}}$  has no action can be captured in the system model of Section II by setting  $B^{nn} = \mathbf{0}$  (if  $n \in \mathcal{N}$ ) or  $B^{m0} = \mathbf{0}$  for all  $m \in \mathcal{N}$  (if  $n = 0$ ),  $R_t^{U^n U^n} = \mathbf{I}$ , and  $R_t^{X U^n}, R_t^{U^n X}, R_t^{U^m U^n}, R_t^{U^n U^m}$  to be zero matrices for all  $m \in \bar{\mathcal{N}} \setminus \{n\}$ . Then, from Theorem 3, the optimal action  $U_t^{n*}$  is zero which means that controller  $n$  takes no action.

2) *Decoupled systems:* Consider the system model of Section II where the dynamics of plant  $n$  in (1) and the instantaneous cost of sub-system  $n$  (that is, plant  $n$  and the local controller  $C^n$  collectively) in (8) are affected only by the  $n$ -th component of the remote controller's action  $U_t^0$ , denoted by  $[U_t^0]_{r,n}$ . Specifically,

$$\begin{aligned} X_{t+1}^n &= A^{nn} X_t^n + B^{nn} U_t^n + \bar{B}^{n0} [U_t^0]_{r,n} + W_t^n, t = 0, \dots, T, \\ c_t(X_t^{1:N}, U_t^{0:N}) &= \sum_{n=1}^N c_t^n(X_t^n, [U_t^0]_{r,n}, U_t^n), \end{aligned} \quad (58)$$

where  $c_t^n$  is a quadratic function of the form (8).

We can still use Theorem 3 to find optimal control strategies in this model. However, it is more efficient to consider the system as consisting of  $N$  decomposed remote controllers  $C^{0n}$ ,  $n \in \mathcal{N}$ , where the remote controller  $C^{0n}$  is associated with only subsystem  $n$ . The problem of finding optimal strategies then decomposes into  $N$  separate problems, each with one remote and one local controller. Each subproblem is a special case of Problem 1. Problems with one local and one remote controller were also investigated in our prior work [1].

## V. EXAMPLE: PLATOON CONTROL

An automated highway system (AHS) is an intelligent transportation system designed for autonomous vehicles. AHS organizes highway traffic into groups of tightly-spaced vehicles called “platoons” to increase capacity, improve road safety, and decrease fuel consumption [41]. Due to the advancement in connected vehicle technology, autonomous vehicles can form platoons by information exchange with Vehicle to Vehicle (V2V) and Vehicle to infrastructure (V2I) communication. In this section, as an example of the system model of Section II, we consider a simple platoon control problem using V2I communication<sup>2</sup>.

Consider a platoon of  $N$  vehicles and an infrastructure as depicted in Fig. 3. Let  $D_t^n$  and  $V_t^n$  denote the location and the velocity of  $n$ -th vehicle at time  $t$ . Then, the location of the  $n$ -th vehicle at time  $t+1$  can be written as  $D_{t+1}^n = D_t^n + V_t^n + W_t^n$  where  $W_t^n$  is a zero-mean random variable with distribution  $\pi_{W_t^n}$  modeling the uncertain road condition. We assume that  $D_0^n$  is a random variable with distribution  $\pi_{D_0^n}$  for  $n \in \mathcal{N}$ . At each time step, each vehicle perfectly observes its location and communicates the location to the infrastructure through an unreliable channel with link failure probability  $p^n$ . The infrastructure broadcasts what it has received from the channels to all vehicles. We assume that the velocity of the first vehicle (the reference vehicle) is fixed to a target velocity  $v$ . Furthermore, each of the follower vehicles  $n = \{2, \dots, N\}$  makes a local decision about its velocity based on its local observation and the information broadcasted from the infrastructure. The objective of the platoon is to keep the distance between every two consecutive vehicles to  $d$  (desired inter-vehicle spacing) while keeping their velocity as close as possible to the target velocity  $v$  for a period of  $T$  time steps. We model the above objective by a cost function  $c_t(D_t^{1:N}, V_t^{1:N})$

$$c_t(D_t^{1:N}, V_t^{1:N}) = \beta \sum_{n=1}^{N-1} (D_t^n - D_t^{n+1} - d)^2 + \sum_{n=2}^N (V_t^n - v)^2, \quad (59)$$

where the first term is the sum of quadratic spacing errors with respect to the desired spacing  $d$ , the second term is the sum of quadratic velocity errors with respect to the target velocity  $v$ ,

<sup>2</sup> Our goal in this section is to illustrate the applicability of our system model, so we only consider V2I communication even though general platoon control can use both V2V and V2I communication. We also note that in more exact formulations of platoon control, each vehicle’s control action is generally a function of its location, velocity, and acceleration [42], [43]. Although these formulations can be captured by the system model of Section II, we consider the velocities as the control actions for the simplicity of presentation.

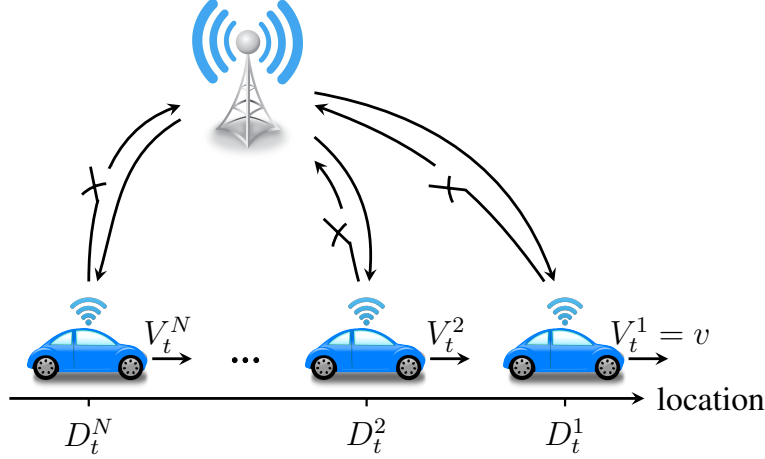


Fig. 3. Platoon control using V2I communication.

and  $\beta$  models the trade-off between these two errors.

This scenario can be described using the system model of Section II by defining  $X_t^n = D_t^n + nd - tv$  and  $U_t^n = V_t^n - v$  for  $n \in \mathcal{N}$ . Then,  $X_{t+1}^n = X_t^n + B^{nn}U_t^n + W_t^n$  where  $B^{11} = 0$  and  $B^{nn} = 1$  for  $n = \{2, \dots, N\}$ ,  $R_t^{UU} = \mathbf{I}_N$ ,  $R_t^{XU} = R_t^{UX} = \mathbf{0}_{N \times N}$ , and

$$R_t^{XX} = \beta \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & \ddots & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}. \quad (60)$$

Now, we can apply Theorem 3 to find the optimal decision  $V_t^{n*}$  for the  $n$ -th vehicle as follows.

**Corollary 1.** *The optimal decisions  $V_t^{n*}$ ,  $n \in \mathcal{N}$  for the vehicles can be computed as follows,*

$$\begin{bmatrix} \bar{V}_t^{1*} \\ \vdots \\ \bar{V}_t^{N*} \end{bmatrix} = -(G_t^{UU})^{-1} G_t^{UX} \begin{bmatrix} \hat{D}_t^1 + d - tv \\ \vdots \\ \hat{D}_t^N + Nd - tv \end{bmatrix} + \begin{bmatrix} v \\ \vdots \\ v \end{bmatrix}, \quad (61)$$

$$V_t^{n*} = \bar{V}_t^{n*} - \left( \tilde{G}_t^{U^n U^n} \right)^{-1} \tilde{G}_t^{U^n X^n} \left( D_t^n - \hat{D}_t^n \right), \quad n \in \mathcal{N} \quad (62)$$

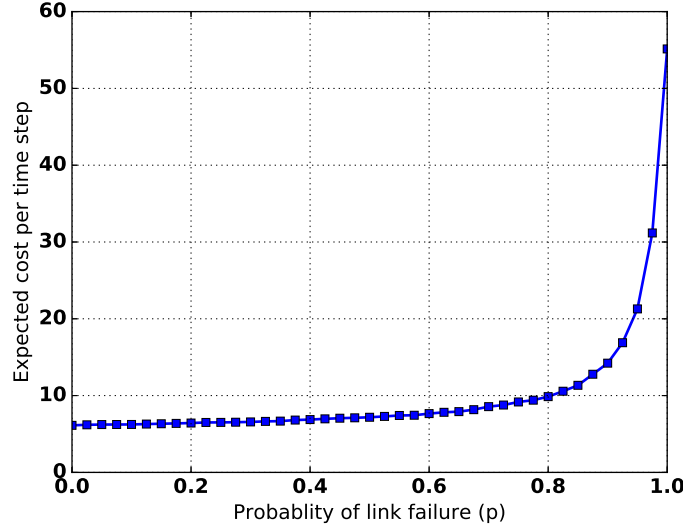


Fig. 4. Expected cost per time for different  $p$

where  $\hat{D}_t^n$  can be computed recursively according to

$$\hat{D}_0^n = \begin{cases} D_0^n & \text{if the infrastructure receives } D_0^n \\ \mu(\pi_{D_0^n}) & \text{otherwise.} \end{cases} \quad (63)$$

$$\hat{D}_{t+1}^n = \begin{cases} D_{t+1}^n & \text{if the infrastructure receives } D_{t+1}^n \\ \hat{D}_t^n + \bar{V}_t^{n*} & \text{otherwise.} \end{cases} \quad (64)$$

We computed the optimal solution for a platoon of  $N = 3$  vehicles with  $v = 5$ ,  $d = 10$ ,  $\pi_{D_0^n} = \rho((3 - n)d)$ ,  $\beta = 1$ , and  $p^n = p$  for  $n = 1, 2, 3$  over a period of  $T = 100$  steps. We assumed that  $W_t^n$ ,  $n = 1, 2, 3$ ,  $t = 0, 1, \dots, T$ , is the standard Gaussian random variable and considered different values of link failure probability  $p$ . Fig. 4 shows the expected cost per time step, that is  $\frac{1}{T+1} \mathbb{E}[\sum_{t=0}^T c_t(D_t^{1:N}, V_t^{1:N*})]$ , for different values of  $p$ . This figure highlights the importance of communication as the expected cost per time step increases significantly with the probability of link failure. It can be seen from this figure that even unreliable V2I links with probability of failure  $p = 0.9$  can remarkably decrease the expected cost compared to the case of no V2I communication (i.e.,  $p = 1$ ).

Fig. 5 shows the velocity of the second and the third vehicles over time for  $p = 0$ ,  $p = 0.5$ ,  $p = 1$ . It can be seen that there is no significant difference in the velocity of the vehicles for different

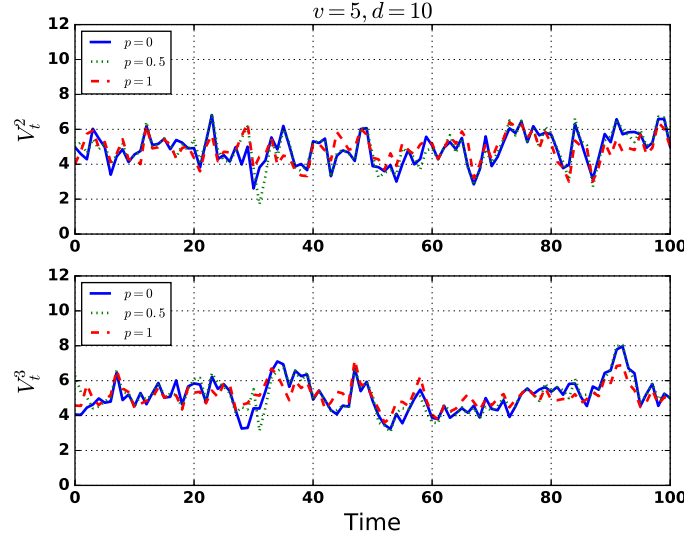


Fig. 5. Velocity of the vehicles per time for  $p = 0, p = 0.5, p = 1$

values of  $p$  and both vehicles can follow the reference vehicle with the velocities around the target value  $v = 5$ .

Fig. 6 indicates the inter-vehicle distances over time for  $p = 0, p = 0.5, p = 1$ . It can be seen that in the case of no communication (i.e.,  $p = 1$ ), the first and the second vehicles collide while the third vehicle can keep its distance from the second car around the desired inter-vehicle distance  $d = 10$  for different values of  $p$ .

## VI. CONCLUSION

We considered a networked control system (NCS) consisting of a remote controller and a collection of linear plants, each associated with a local controller. Each local controller directly observes the state of its co-located plant and can inform the remote controller of the plant's state through an unreliable uplink channel. The downlink channels from the remote controller to local controllers are assumed to be perfect. The objective of the local controllers and the remote controller is to cooperatively minimize a quadratic performance cost. This multi-controller NCS problem is not a partially nested LQG problem, hence we cannot directly use prior results in decentralized control to conclude that linear strategies are optimal.

We employed the common information approach to this problem and showed that it is equiva-

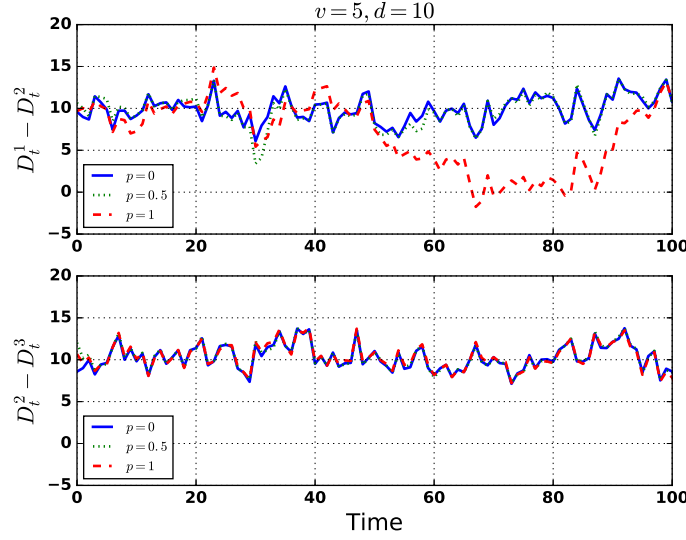


Fig. 6. Inter-vehicle distances per time for  $p = 0, p = 0.5, p = 1$

lent to a centralized sequential decision-making problem where the remote controller is the only decision-maker. We provided a dynamic program to obtain optimal strategies in the equivalent problem. Then, using these optimal strategies, we obtained optimal control strategies for all local controllers and the remote controller in our original problem. In the optimal control strategies, all controllers compute common estimates of the states of the plants based on the common information from the communication network. The remote controller's action is linear in the common estimated states, and the action of each local controller is linear in both the actual state of its corresponding plant and the common estimated states.

We applied the results to a simple platoon control problem for autonomous vehicles. The example showed the importance of communication in NCS, and it also demonstrated the effect of unreliable communication on system performance.

Our results sketch a solution methodology for decentralized control with unreliable communication among the controllers. The methodology can potentially be applied to other communication topologies in decentralized control such as broadcast networks and acyclic communication networks with unreliable channels.

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## APPENDIX

In the Appendices, we use  $\{X^m\}$  to denote  $\{X^m\}_{m \in \mathcal{A}}$  if the set  $\mathcal{A}$  is clear from the context.

## APPENDIX A

### PRELIMINARY RESULTS

In this section, we state and prove a set of claims which are useful in proving the main results of this paper.

**Claim 1.** *Let  $\mathcal{F}^0$ ,  $\mathcal{F}^{1:N}$  and  $\mathcal{G}^{1:N}$  be  $\sigma$ -algebras such that  $\mathcal{F}^{1:N}$  are conditionally independent given  $\mathcal{F}^0$ , and  $\mathcal{G}^n \subset \mathcal{F}^n$ ,  $n \in \mathcal{N}$ . Then, for  $A^n \in \mathcal{F}^n$ ,  $n \in \mathcal{N}$ ,*

$$\mathbb{P}(\{A^n\} | \mathcal{F}^0, \{\mathcal{G}^m\}) = \prod_{n \in \mathcal{N}} \mathbb{P}(A^n | \mathcal{F}^0, \{\mathcal{G}^m\}). \quad (65)$$

*Proof:* Showing the correctness of (65) is the same as showing

$$\mathbb{E}[\prod_{n \in \mathcal{N}} \mathbb{1}_{A^n} | \mathcal{F}^0, \{\mathcal{G}^m\}] = \prod_{n \in \mathcal{N}} \mathbb{E}[\mathbb{1}_{A^n} | \mathcal{F}^0, \{\mathcal{G}^m\}]. \quad (66)$$

The left hand side of (66) can be written as,

$$\begin{aligned}
& \mathbb{E} \left[ \prod_{n \in \mathcal{N}} \mathbb{1}_{A^n} \middle| \mathcal{F}^0, \{\mathcal{G}^m\} \right] \\
&= \mathbb{E} \left[ \mathbb{E} \left[ \prod_{n \in \mathcal{N}} \mathbb{1}_{A^n} \middle| \mathcal{F}^0, \mathcal{G}^k, \{\mathcal{F}^m\}_{m \neq k} \right] \middle| \mathcal{F}^0, \{\mathcal{G}^m\} \right] \\
&= \mathbb{E} \left[ \prod_{n \neq k} \mathbb{1}_{A^n} \mathbb{E}[\mathbb{1}_{A^k} \middle| \mathcal{F}^0, \mathcal{G}^k, \{\mathcal{F}^m\}_{m \neq k}] \middle| \mathcal{F}^0, \{\mathcal{G}^m\} \right] \\
&= \mathbb{E} \left[ \prod_{n \neq k} \mathbb{1}_{A^n} \mathbb{E}[\mathbb{1}_{A^k} \middle| \mathcal{F}^0, \mathcal{G}^k] \middle| \mathcal{F}^0, \{\mathcal{G}^m\} \right] \\
&= \mathbb{E} \left[ \prod_{n \neq k} \mathbb{1}_{A^n} \middle| \mathcal{F}^0, \{\mathcal{G}^m\} \right] \mathbb{E}[\mathbb{1}_{A^k} \middle| \mathcal{F}^0, \mathcal{G}^k] \tag{67}
\end{aligned}$$

where the first equality is true due to tower property, the second property is true due to “pulling out known factors” property, the third equality is true due to Doob’s conditional independence property and “chain rule” property [44, Chapter 5], and the fourth equality is true again due to “pulling out known factors” property.

By repeating the procedure of (67) one by one for each  $k \in \mathcal{N}$ , then we get

$$\begin{aligned}
& \mathbb{E} \left[ \prod_{n \neq k} \mathbb{1}_{A^n} \middle| \mathcal{F}^0, \{\mathcal{G}^m\} \right] \mathbb{E}[\mathbb{1}_{A^k} \middle| \mathcal{F}^0, \mathcal{G}^k] \\
&= \prod_{k \in \mathcal{N}} \mathbb{E}[\mathbb{1}_{A^k} \middle| \mathcal{F}^0, \mathcal{G}^k] = \prod_{n \in \mathcal{N}} \mathbb{E}[\mathbb{1}_{A^n} \middle| \mathcal{F}^0, \{\mathcal{G}^m\}] \tag{68}
\end{aligned}$$

where last equality is true due to Doob’s conditional independence property and “chain rule” property. ■

**Claim 2.** 1) Consider feasible strategies  $g = g^{0:N}$ ,  $g^n \in \mathcal{G}^n$ ,  $n \in \mathcal{N}$ , in Problem 1. Then, the random vectors  $X_{0:t}^n$  are conditionally independent of  $X_{0:t}^m$  for  $n, m \in \mathcal{N}$ ,  $n \neq m$  given  $H_t^0$ . That is, for any measurable sets  $E_{0:t}^n \subset \prod_{s=0}^t \mathbb{R}^{d_x^n}$ ,  $n \in \mathcal{N}$ ,

$$\mathbb{P}^g(\{X_{0:t}^n \in E_{0:t}^n\} | H_t^0) = \prod_{n \in \mathcal{N}} \mathbb{P}^g(X_{0:t}^n \in E_{0:t}^n | H_t^0). \tag{69}$$

2) The same result holds under any feasible fixed prescription strategy  $\phi^{prs} \in \Phi^{prs}$  in Problem 2.

*Proof:* We prove (69) by induction. At time 0, (69) is true because random vectors  $X_0^{1:N}$  are independent. Suppose (69) is true at time  $t$ .

At time  $t + 1$ , for all  $n \in \mathcal{N}$ , define

$$\begin{aligned}
\tilde{f}_t^n(X_{0:t}^n, W_t^n, H_t^0) &= X_{t+1}^n \\
&= A^{nn}X_t^n + B^{nn}g_t^n(X_{0:t}^n, H_t^0, U_{0:t-1}^n) + B^{n0}g_t^0(H_t^0) + W_t^n \\
&= A^{nn}X_t^n + B^{nn}\tilde{g}_t^n(X_{0:t}^n, H_t^0) + B^{n0}g_t^0(H_t^0) + W_t^n
\end{aligned} \tag{70}$$

where  $\tilde{g}_t^n$  is obtained from  $g_{0:t}^n$  by recursively substituting for  $U_{0:t-1}^n$ .

Then, the left hand side of (69) at time  $t + 1$  becomes

$$\begin{aligned}
&\mathbb{P}^g \left( \{X_{0:t+1}^n \in E_{0:t+1}^n\} | H_{t+1}^0 \right) \\
&= \mathbb{P}^g \left( \{X_{0:t+1}^n \in E_{0:t+1}^n\} \middle| H_t^0, \{\Gamma_{t+1}^n, \Gamma_{t+1}^n X_{t+1}^n\} \right) \\
&= \mathbb{P}^g \left( \{\tilde{f}_t^n(X_{0:t}^n, W_t^n, H_t^0) \in E_{t+1}^n, X_{0:t}^n \in E_{0:t}^n\} \right. \\
&\quad \left. \middle| H_t^0, \{\Gamma_{t+1}^n, \Gamma_{t+1}^n \tilde{f}_t^n(X_{0:t}^n, W_t^n, H_t^0)\} \right).
\end{aligned} \tag{71}$$

Note that 1)  $\Gamma_{t+1}^{1:N}, W_t^{1:N}$  are independent of all other variables at time  $t$ , and 2)  $X_{0:t}^{1:N}$  are independent conditioned on  $H_t^0$  from the induction hypothesis. Hence, if we define  $\mathcal{F}^0 = \sigma(H_t^0, \{\Gamma_{t+1}^n\})$ ,  $\mathcal{F}^n = \sigma(X_{0:t}^n, W_t^n, H_t^0, \Gamma_{t+1}^n)$ , and  $\mathcal{G}^n = \sigma(\Gamma_{t+1}^n \tilde{f}_t^n(X_{0:t}^n, W_t^n, H_t^0))$ , then according to Claim 1,

$$\begin{aligned}
&\mathbb{P}^g \left( \{\tilde{f}_t^n(X_{0:t}^n, W_t^n, H_t^0) \in E_{t+1}^n, X_{0:t}^n \in E_{0:t}^n\} \right. \\
&\quad \left. \middle| H_t^0, \{\Gamma_{t+1}^n, \Gamma_{t+1}^n \tilde{f}_t^n(X_{0:t}^n, W_t^n, H_t^0)\} \right) \\
&= \prod_{n \in \mathcal{N}} \mathbb{P}^g \left( \tilde{f}_t^n(X_{0:t}^n, W_t^n, H_t^0) \in E_{t+1}^n, X_{0:t}^n \in E_{0:t}^n \right. \\
&\quad \left. \middle| H_t^0, \{\Gamma_{t+1}^n, \Gamma_{t+1}^n \tilde{f}_t^n(X_{0:t}^n, W_t^n, H_t^0)\} \right) \\
&= \prod_{n \in \mathcal{N}} \mathbb{P}^g \left( X_{0:t+1}^n \in E_{0:t+1}^n | H_{t+1}^0 \right).
\end{aligned} \tag{72}$$

Therefore, (69) is true at time  $t$  and the proof of the first part is complete.

The second part can be proved in a similar way. ■

**Corollary 2.** *For any feasible prescription strategy  $\phi^{prs} \in \Phi^{prs}$  in Problem 2 ( $g = g^{0:N}$ ,  $g^n \in \mathcal{G}^n$ ,  $n \in \mathcal{N}$ , in Problem 1),  $(X_{t+1}^n, Z_{t+1}^n)$  is conditionally independent of  $\{Z_{t+1}^m\}_{m \neq n}$  given any realization  $H_t^0$ .*

**Claim 3.** 1) For any constant vector  $x \in \prod_{n=1}^N \mathbb{R}^{d_X^n}$ ,

$$\min_{\{u^n \in \mathbb{R}^{d_U^n}\}_{n \in \mathcal{N}}} QF(R_t, \text{vec}(x, \{u^n\}_{n \in \mathcal{N}})) = QF(P_t, x) \quad (73)$$

where  $P_t := R_t^{XX} - R_t^{XU} (R_t^{UU})^{-1} R_t^{UX}$  is the Schur complement of  $R_t^{UU}$  of  $R_t$  and the optimal solution is given by,

$$\text{vec}(\{u^{n*}\}_{n \in \mathcal{N}}) = - (R_t^{UU})^{-1} R_t^{UX} x. \quad (74)$$

2) For any  $\theta^{1:N} \in \prod_{n=1}^N \Delta(\mathbb{R}^{d_X^n})$ , let  $X^{\theta^1}, \dots, X^{\theta^N}$  be independent random variables such that  $X^{\theta^n}$  has distribution  $\theta^n, n \in \mathcal{N}$ . Then

$$\begin{aligned} & \min_{\{q^n \in \mathcal{Q}^n(\theta^n)\}_{n \in \mathcal{N}}} \\ & \text{tr} \left( R_t \text{cov} \left( \text{vec}(\{X^{\theta^n}\}_{n \in \mathcal{N}}, 0, \{q^n(X^{\theta^n})\}_{n \in \mathcal{N}}) \right) \right) \\ & = \sum_{n=1}^N \text{tr} \left( P_t^{nn} \text{cov}(X^{\theta^n}) \right) \end{aligned} \quad (75)$$

where

$$P_t^{nn} := R_t^{X^n X^n} - R_t^{X^n U^n} (R_t^{U^n U^n})^{-1} R_t^{U^n X^n} \quad (76)$$

and the optimal solution for  $n \in \mathcal{N}$  is given by,

$$q^{n*}(X^{\theta^n}) = - (R_t^{U^n U^n})^{-1} R_t^{U^n X^n} (X^{\theta^n} - \mu(\theta^n)). \quad (77)$$

*Proof:* The first part of Claim 3 can be obtained by a simple completing the square argument.

Now let's consider the functional optimization problem (75) in the second part of Claim 3.

Using properties of trace and covariance matrices, we can write

$$\begin{aligned} & \text{tr} \left( R_t \text{cov} \left( \text{vec}(\{X^{\theta^n}\}_{n \in \mathcal{N}}, 0, \{q^n(X^{\theta^n})\}_{n \in \mathcal{N}}) \right) \right) \\ & = \mathbb{E} \left[ QF \left( R_t, \text{vec}(\{X^{\theta^n}\}_{n \in \mathcal{N}}, 0, \{q^n(X^{\theta^n})\}_{n \in \mathcal{N}}) \right) \right. \\ & \quad \left. - \mathbb{E}[\text{vec}(\{X^{\theta^n}\}_{n \in \mathcal{N}}, 0, \{q^n(X^{\theta^n})\}_{n \in \mathcal{N}})] \right] \\ & = \mathbb{E} \left[ QF \left( R_t, \text{vec}(\{X^{\theta^n} - \mu(\theta^n)\}_{n \in \mathcal{N}}, 0, \{q^n(X^{\theta^n})\}_{n \in \mathcal{N}}) \right) \right] \\ & = \sum_{n \in \mathcal{N}} \mathbb{E} \left[ QF \left( \tilde{R}_t^n, \text{vec} \left( X^{\theta^n} - \mu(\theta^n), q^n(X^{\theta^n}) \right) \right) \right] \end{aligned} \quad (78)$$

where  $\tilde{R}_t^n = \begin{bmatrix} R_t^{X^n X^n} & R_t^{X^n U^n} \\ R_t^{U^n X^n} & R_t^{U^n U^n} \end{bmatrix}$ . The last equality in (78) is true because all off-diagonal terms are zero since  $\mathbb{E}[q^n(X^{\theta^n})] = 0$ ,  $\mathbb{E}[X^{\theta^n} - \mu(\theta^n)] = 0$ , and  $X^{\theta^n}$  and  $X^{\theta^m}$  are independent for all  $n \neq m$ .

Note that each term in (78) only depends on one  $q^n$ ,  $n \in \mathcal{N}$ . Therefore, the functional optimization problem (75) is equivalent to solving the  $N$  optimization problems

$$\min_{q^n \in \mathcal{Q}^n(\theta^n)} \mathbb{E} \left[ QF \left( \tilde{R}_t^n, \text{vec} \left( X^{\theta^n} - \mu(\theta^n), q^n(X^{\theta^n}) \right) \right) \right]. \quad (79)$$

Since  $\theta^n$  is the distribution of  $X^{\theta^n}$ , we have

$$\begin{aligned} & \mathbb{E} \left[ QF \left( \tilde{R}_t^n, \text{vec} \left( X^{\theta^n} - \mu(\theta^n), q^n(X^{\theta^n}) \right) \right) \right] \\ &= \int QF \left( \tilde{R}_t^n, \text{vec} \left( y - \mu(\theta^n), q^n(y) \right) \right) \theta^n(dy). \end{aligned} \quad (80)$$

Note that the function inside the integral of (80) is a quadratic function. As in the first part of Claim 3, for any  $y \in \mathbb{R}^{d_X^n}$  we have

$$\begin{aligned} & QF \left( \tilde{R}_t^n, \text{vec} \left( y - \mu(\theta^n), q^n(y) \right) \right) \geq \\ & QF \left( \tilde{R}_t^n, \text{vec} \left( y - \mu(\theta^n), q^{n*}(y) \right) \right) = QF \left( P_t^{nn}, y - \mu(\theta^n) \right) \end{aligned}$$

where  $P_t^{nn}$  is given by (76) and  $q^{n*}$  is the function given by (77). Note that  $q^{n*} \in \mathcal{Q}^{\theta^n}$  because  $q^{n*}$  is measurable and

$$\begin{aligned} & \int q^{n*}(x^n) \theta^n(dx^n) = \\ & \int - (R_t^{U^n U^n})^{-1} R_t^{U^n X^n} (x^n - \mu(\theta^n)) \theta^n(dx^n) = 0. \end{aligned}$$

Thus  $q^{n*}$  is the optimal solution for the optimization problem in (79) for each  $n \in \mathcal{N}$  and the optimal value is  $\text{tr}(P_t^{nn} \text{cov}(X^{\theta^n}))$ . Using (78), the optimal value in (75) then becomes  $\sum_{n=1}^N \text{tr} (P_t^{nn} \text{cov}(X^{\theta^n}))$ . ■

## APPENDIX B

### PROOF OF LEMMA 1

To show that the local controller  $C^n$ ,  $n \in \mathcal{N}$ , can use only  $\hat{H}_t^n = \{X_t^n, H_t^0\}$  to make the decision at time  $t$  without loss of optimality, we proceed using *person-by-person* approach.

For any fixed feasible strategies of the remote controller  $g^0$  and the local controllers  $g^m$ ,  $m \in \mathcal{N} \setminus \{n\}$ , the problem of finding optimal strategy of the local controller  $n$  becomes a centralized problem with the state  $\hat{X}_t = \text{vec}(X_{0:t-1}^{-n}, X_t^{1:N}, H_t^0)$ . From the theory of centralized control problems with imperfect information [45], we know that we can restrict controller  $C^n$ 's strategy to be of the form:

$$U_t^{n*} = \sigma_t^n(\mathbb{P}^{g^0, g^{-n}}(\hat{X}_t | H_t^n)). \quad (81)$$

Then, if we denote  $\tilde{g} = (g^0, g^{-n})$ , for any measurable sets  $F \subset \mathcal{H}_t^0$ ,  $E_s^m \in \mathbb{R}^{d_X^m}$ ,  $m \in \mathcal{N} \setminus \{n\}$ ,  $s = 0, 1, \dots, t-1$ , and  $E_t^m \in \mathbb{R}^{d_X^m}$ ,  $m \in \mathcal{N}$ ,

$$\begin{aligned} & \mathbb{P}^{\tilde{g}}(\hat{X}_t \in \text{vec}(E_{0:t-1}^{-n}, E_t^{1:N}, F) | H_t^n) \\ &= \mathbb{P}^{\tilde{g}}(X_{0:t}^{-n} \in E_{0:t}^{-n}, X_t^n \in E_t^n, H_t^0 \in F | X_{0:t}^n, H_t^0) \\ &= \mathbb{1}_F(H_t^0) \mathbb{P}^{\tilde{g}}(X_{0:t}^{-n} \in E_{0:t}^{-n}, X_t^n \in E_t^n | X_{0:t}^n, H_t^0) \\ &= \mathbb{1}_F(H_t^0) \mathbb{P}^{\tilde{g}}(X_t^n \in E_t^n | X_t^n, H_t^0) \prod_{m \neq n} \mathbb{P}^{\tilde{g}}(X_{0:t}^m \in E_{0:t}^m | H_t^0) \\ &= \mathbb{P}^{\tilde{g}}(\hat{X}_t \in \text{vec}(E_{0:t-1}^{-n}, E_t^{1:N}, F) | X_t^n, H_t^0) \end{aligned} \quad (82)$$

where the second equality is true due to the “pulling out known factors” property, the third equality is true from Claim 2, and the last equality follows from the same reasons as the first three equalities. Therefore, the local controller  $C^n$  can use only  $\hat{H}_t^n = \{X_t^n, H_t^0\}$  to make the optimal decision at time  $t$ .

*An alternative proof based on Markov decision Problem (MDP):*

To show that the local controller  $C^n$ ,  $n \in \mathcal{N}$ , can use only  $\hat{H}_t^n = \{X_t^n, H_t^0\}$  to make the decision at time  $t$  without loss of optimality, we proceed using *person-by-person* approach. For any fixed feasible strategies of the remote controller  $g^0$  and the local controllers  $g^m$ ,  $m \in \mathcal{N} \setminus \{n\}$ , the problem of finding optimal strategy of the local controller  $n$  can be reduced to a Markov decision problem with  $(X_t^n, H_t^0)$  as the (perfectly observed) state—In particular, it can be shown that this state evolves in a controlled Markovian fashion with  $U_t^n$  as the control action. Moreover, by averaging over  $X_{0:t}^{-n}$ , the expected cost at time  $t$  can be written as a function of this state and the action  $U_t^n$ . From the theory of centralized control problems with perfect state information [45], we know that we can restrict to control strategies for  $C^n$  that are of the form:

$$U_t^{n*} = \sigma_t^n(X_t^n, H_t^0). \quad (83)$$

## APPENDIX C

## PROOF OF LEMMA 2

If  $g = g^{0:N}$  where  $g^0 \in \mathcal{G}^0$  and  $g^n \in \hat{\mathcal{G}}^n$ ,  $n \in \mathcal{N}$ , are control strategies in Problem 1, define  $\phi_t^0$ ,  $\bar{\phi}_t^n$  and  $\tilde{\phi}_t^n$  according to (21)-(23).

Since  $\phi_t^0(H_t^0) = g_t^0(H_t^0) \in \mathcal{G}^0$  and  $\phi_t^n(X_t^n, H_t^0) = \bar{\phi}_t^n(H_t^0) + [\tilde{\phi}_t^n(H_t^0)](X_t^n) = g_t^n(X_t^n, H_t^0) \in \hat{\mathcal{G}}^n$ , conditions (C1) and (C2) for feasible prescription strategies are satisfied. Furthermore,

$$\begin{aligned} \mathbb{E}^{\phi^{prs}} [\tilde{\phi}_t^n(H_t^0)](X_t^n | H_t^0) &= \mathbb{E}^g [\tilde{\phi}_t^n(H_t^0)](X_t^n | H_t^0) \\ &= \mathbb{E}^g [g_t^n(X_t^n, H_t^0) | H_t^0] - \mathbb{E}^g [g_t^n(X_t^n, H_t^0) | H_t^0] = 0, \end{aligned} \quad (84)$$

so  $\phi^{prs}$  satisfies all three conditions (C1)-(C3) and hence,  $\phi^{prs} \in \Phi^{prs}$ . By definition, the control actions  $U_t^0$  and  $U_t^n$ ,  $n \in \mathcal{N}$ , under  $\phi^{prs}$  are the same as these control actions under  $g$ . Hence, they generate the same probability measure on the state and the control actions, and consequently,

$$\begin{aligned} \lambda(\phi^{prs}) &= \mathbb{E}^{\phi^{prs}} \left[ \sum_{t=0}^T c_t(X_t^{1:N}, U_t^{0:N}) \right] \\ &= \mathbb{E}^g \left[ \sum_{t=0}^T c_t(X_t^{1:N}, U_t^{0:N}) \right] = J(g). \end{aligned} \quad (85)$$

On the other hand, if  $\phi^{prs} \in \Phi^{prs}$  is a prescription strategy in Problem 2, then  $g_t^0$  and  $g_t^{1:N}$  defined by (25) and (26) are valid control strategies for Problem 1, that is,  $g_t^0 \in \mathcal{G}^0$  and  $g_t^n \in \hat{\mathcal{G}}^n$ ,  $n \in \mathcal{N}$ . Furthermore, similar to (85), we have  $J(g) = \Lambda(\phi^{prs})$ .

## APPENDIX D

## PROOF OF LEMMA 4

Note that from (31)-(35),  $[\nu_t^n(\cdot)](E^n) : \mathcal{H}_t^0 \mapsto \mathbb{R}$  is a measurable function. To show that  $[\nu_t^n(H_t^0)](E^n) = \mathbb{P}^{\phi_{0:t-1}^{prs}}(X_t^n \in E^n | H_t^0)$ , first note that for any  $t$

$$\begin{aligned} \mathbb{P}^{\phi_{0:t-1}^{prs}}(X_t^n \in E^n | H_t^0) &= \mathbb{P}^{\phi_{0:t-1}^{prs}}(X_t^n \in E^n | H_{t-1}^0, \{Z_t^m\}) \\ &= \mathbb{P}^{\phi_{0:t-1}^{prs}}(X_t^n \in E^n | H_{t-1}^0, Z_t^n), \end{aligned} \quad (86)$$

where the second equality is true because of Corollary 2.

We now prove by induction that

$$[\nu_t^n(H_t^0)](E^n) = \mathbb{P}^{\phi_{0:t-1}^{prs}}(X_t^n \in E^n | H_{t-1}^0, Z_t^n). \quad (87)$$

At time  $t = 0$ , since  $\Gamma_0^n \in \{0, 1\}$ , consider two cases:

- If  $\Gamma_0^n = 1$ ,

$$\begin{aligned}\mathbb{P}(X_0^n \in E^n | Z_0^n) \mathbb{1}_{\{\Gamma_0^n=1\}} &= \mathbb{P}(X_0^n \in E^n | X_0^n, \Gamma_0^n) \mathbb{1}_{\{\Gamma_0^n=1\}} \\ &= \mathbb{P}(X_0^n \in E^n | X_0^n) \mathbb{1}_{\{\Gamma_0^n=1\}} = \mathbb{1}_{E^n}(X_0^n) \mathbb{1}_{\{\Gamma_0^n=1\}}.\end{aligned}\quad (88)$$

- If  $\Gamma_0^n = 0$ ,

$$\begin{aligned}\mathbb{P}(X_0^n \in E^n | Z_0^n) \mathbb{1}_{\{\Gamma_0^n=0\}} &= \mathbb{P}(X_0^n \in E^n | \Gamma_0^n) \mathbb{1}_{\{\Gamma_0^n=0\}} \\ &= \mathbb{P}(X_0^n \in E^n) \mathbb{1}_{\{\Gamma_0^n=0\}} = \pi_{X_0^n}(E^n) \mathbb{1}_{\{\Gamma_0^n=0\}}.\end{aligned}\quad (89)$$

Hence, (87) holds at time 0. Assume that (87) holds at time  $t$ . This means that  $\mathbb{P}^{\phi_{0:t-1}^{prs}}(dx_t^n | H_t^0) = [\nu_t^n(H_t^0)](dx_t^n)$  and since  $W_t^n$  is independent of all random variables at and before time  $t$ , we get

$$\mathbb{P}^{\phi_{0:t-1}^{prs}}(dx_t^n dw_t^n | H_t^0) = [\nu_t^n(H_t^0)](dx_t^n) \pi_{W_t^n}(dw_t^n). \quad (90)$$

At time  $t + 1$ , since  $\Gamma_{t+1}^n \in \{0, 1\}$ , consider two cases:

- If  $\Gamma_{t+1}^n = 1$ , similar to (88) we obtain

$$\begin{aligned}\mathbb{P}^{\phi_{0:t}^{prs}}(X_{t+1}^n \in E^n | H_t^0, Z_{t+1}^n) \mathbb{1}_{\{\Gamma_{t+1}^n=1\}} \\ = \mathbb{1}_{E^n}(X_{t+1}^n) \mathbb{1}_{\{\Gamma_{t+1}^n=1\}} = [\nu_{t+1}^n(H_{t+1}^0)](E^n) \mathbb{1}_{\{\Gamma_{t+1}^n=1\}}.\end{aligned}\quad (91)$$

- If  $\Gamma_{t+1}^n = 0$ ,

$$\begin{aligned}\mathbb{P}^{\phi_{0:t}^{prs}}(X_{t+1}^n \in E^n | H_t^0, Z_{t+1}^n) \mathbb{1}_{\{\Gamma_{t+1}^n=0\}} \\ = \mathbb{P}^{\phi_{0:t}^{prs}}(X_{t+1}^n \in E^n | H_t^0, \Gamma_{t+1}^n) \mathbb{1}_{\{\Gamma_{t+1}^n=0\}} \\ = \mathbb{P}^{\phi_{0:t}^{prs}}(f_{t+1}^n(X_t^n, W_t^n, \phi_t^{prs}(H_t^0)) \in E^n | H_t^0) \mathbb{1}_{\{\Gamma_{t+1}^n=0\}} \\ = \mathbb{E}^{\phi_{0:t}^{prs}}[\mathbb{1}_{E^n}(f_{t+1}^n(X_t^n, W_t^n, \phi_t^{prs}(H_t^0))) | H_t^0] \mathbb{1}_{\{\Gamma_{t+1}^n=0\}} \\ = \int \int \mathbb{1}_{E^n}(f_{t+1}^n(x_t^n, w_t^n, \phi_t^{prs}(H_t^0))) \\ \mathbb{P}^{\phi_{0:t-1}^{prs}}(dx_t^n dw_t^n | H_t^0) \mathbb{1}_{\{\Gamma_{t+1}^n=0\}} \\ = \int \int \mathbb{1}_{E^n}(f_{t+1}^n(x_t^n, w_t^n, \phi_t^{prs}(H_t^0))) \\ \nu_t^n(H_t^0)(dx_t^n) \pi_{W_t^n}(dw_t^n) \mathbb{1}_{\{\Gamma_{t+1}^n=0\}} \\ = [\nu_{t+1}^n(H_{t+1}^0)](E^n) \mathbb{1}_{\{\Gamma_{t+1}^n=0\}},\end{aligned}\quad (92)$$



where the second equality is true due to (35) and the fact that  $\Gamma_{t+1}^n$  is independent of  $X_{t+1}^n$  and  $H_t^0$ , the forth equality is true due to the disintegration theorem [44], and the fifth equality is true due to (90).

Hence, (87) holds at time  $t + 1$  and the proof of Lemma 4 is complete.

## APPENDIX E

### PROOF OF THEOREM 1

For any  $\phi^{prs} \in \Phi^{prs}$  and any realization  $h_t^0 \in \mathcal{H}_t^0$ , let the realization of the common belief  $\Theta_t^n$  be  $\theta_t^n = \nu_t^n(h_t^0)$ ,  $n \in \mathcal{N}$ , defined by Lemma 4. Suppose the prescription strategy  $\phi^{prs*} \in \Phi^{prs}$  achieves the minimum of (39) for  $\theta_t^{1:N}$ ,  $t = 0, \dots, T$ , and let  $u_t^{prs*} = (u_t^{0*}, \bar{u}_t^{1:N*}, q_t^{1:N*}) = \phi^{prs*}(h_t^0)$  for any realization  $h_t^0 \in \mathcal{H}_t^0$ .

We prove by induction that  $V_t(\{\nu_t^n(h_t^0)\})$  is a measurable function with respect to  $h_t^0$ , and for any  $h_t^0 \in \mathcal{H}_t^0$  we have

$$\begin{aligned} & \mathbb{E}^{\phi'_t} \left[ \sum_{s=t}^T c_s^{prs}(X_s^{1:N}, U_s^{prs}) \middle| h_t^0 \right] \\ &= V_t(\{\nu_t^n(h_t^0)\}) \end{aligned} \tag{93}$$

$$\leq \mathbb{E}^{\phi^{prs}} \left[ \sum_{s=t}^T c_s^{prs}(X_s^{1:N}, U_s^{prs}) \middle| h_t^0 \right] \tag{94}$$

where  $\phi'_t = \{\phi_{0:t-1}^{prs}, \phi_{t:T}^{prs*}\}$ . Note that the above equation at  $t = 0$  gives the optimality of  $\phi^{prs*}$  for Problem 2.

We first consider (93). At  $T + 1$ , (93) is true (all terms are defined to be 0 at  $T + 1$ ). Assume  $V_{t+1}(\{\nu_{t+1}^n(h_{t+1}^0)\})$  is a measurable function with respect to  $h_{t+1}^0$  and (93) is true at  $t + 1$ .

From the tower property of conditional expectation we have

$$\begin{aligned} & \mathbb{E}^{\phi'_t} \left[ \sum_{s=t}^T c_s^{prs}(X_s^{1:N}, U_s^{prs}) \middle| h_t^0 \right] \\ &= \mathbb{E}^{\phi'_t} [c_t^{prs}(X_t^{1:N}, U_t^{prs}) | h_t^0] \\ &+ \mathbb{E}^{\phi'_t} \left[ \mathbb{E}^{\phi'_t} \left[ \sum_{s=t+1}^T c_s^{prs}(X_s^{1:N}, U_s^{prs}) \middle| H_{t+1}^0 \right] \middle| h_t^0 \right]. \end{aligned} \tag{95}$$

Note that the first term in (95) is equal to

$$\int c_t^{prs}(x_t^{1:N}, u_t^{prs*}) \prod_{n \in \mathcal{N}} \theta_t^n(dx_t^n) = IC(\theta_t^{1:N}, u_t^{prs*}). \quad (96)$$

From the induction hypothesis,  $V_{t+1}(\{\nu_{t+1}^n(h_{t+1}^0)\})$  is measurable with respect to  $h_{t+1}^0$ , and (93) holds at  $t+1$ . Since  $\nu_{t+1}^n(h_{t+1}^0) = \psi_t^n(\theta_t^n, u_t^{prs*}, z_{t+1}^n)$ , the second term in (95) can be written as

$$\begin{aligned} & \mathbb{E}^{\phi'_t} \left[ V_{t+1}(\{\nu_{t+1}^n(H_{t+1}^0)\}) \middle| h_t^0 \right] \\ &= \mathbb{E}^{\phi'_t} \left[ V_{t+1}(\{\psi_t^n(\theta_t^n, u_t^{prs*}, Z_{t+1}^n)\}) \middle| h_t^0 \right] \\ &= \sum_{\gamma_{t+1}^1 \in \{0,1\}} \dots \sum_{\gamma_{t+1}^N \in \{0,1\}} \prod_{n=1}^N LS(p_n, \gamma_{t+1}^n) \times \\ & \mathbb{E}^{\phi'_t} \left[ V_{t+1}(\{\psi_t^n(\theta_t^n, u_t^{prs*}, Z_{t+1}^n)\}) \middle| h_t^0, \{\Gamma_{t+1}^n = \gamma_{t+1}^n\} \right] \end{aligned} \quad (97)$$

where (97) follows from the fact that  $\mathbb{P}(\Gamma_{t+1}^n = 0) = 1 - \mathbb{P}(\Gamma_{t+1}^n = 1) = p_n$ . From Lemma 4 we have

$$\begin{aligned} \psi_t^n(\theta_t^n, u_t^{prs*}, Z_{t+1}^n) &= (1 - \Gamma_{t+1}^n) \alpha_t^{n*} + \Gamma_{t+1}^n \rho(X_{t+1}^n) \\ &= NB(\Gamma_{t+1}^n, \alpha_t^{n*}, X_{t+1}^n) \end{aligned} \quad (98)$$

where  $\alpha_t^{n*} = \psi_t^n(\theta_t^n, u_t^{prs*}, \emptyset)$ . Consequently, each inner term in (97) can be written as

$$\begin{aligned} & \mathbb{E}^{\phi'_t} \left[ V_{t+1}(\{NB(\Gamma_{t+1}^n, \alpha_t^{n*}, X_{t+1}^n)\}) \middle| h_t^0, \{\Gamma_{t+1}^n = \gamma_{t+1}^n\} \right] \\ &= \mathbb{E}^{\phi'_t} \left[ V_{t+1}(\{NB(\gamma_{t+1}^n, \alpha_t^{n*}, X_{t+1}^n)\}) \middle| h_t^0, \{\Gamma_{t+1}^n = 0\} \right] \\ &= \int V_{t+1}(\{NB(\gamma_{t+1}^n, \alpha_t^{n*}, x_{t+1}^n)\}) \prod_{n \in \mathcal{N}} \alpha_t^n(dx_{t+1}^n). \end{aligned} \quad (99)$$

The first equality in (99) is true because  $X_{t+1}^{1:N}$  are independent of  $\Gamma_{t+1}^{1:N}$  and the last equality in (99) follows from Lemma 4.

Combining (96), (97), and (99), the right hand side of (95) is  $V_t(\theta_t^{1:N})$  from the definition of the value function (39) which is equal to  $V_t(\{\nu_t^n(h_t^0)\})$ . Hence, (93) is true at time  $t$ . The measurability of  $V_t(\{\nu_t^n(h_t^0)\})$  with respect to  $h_t^0$  is also resulted from the fact that  $V_t(\{\nu_t^n(h_t^0)\})$  is equal to the conditional expectation  $\mathbb{E}^{\phi'_t} \left[ \sum_{s=t}^T c_s^{prs}(X_s^{1:N}, U_s^{prs}) \middle| h_t^0 \right]$  which is measurable with respect to  $h_t^0$ .

Now let's consider (94). At  $T + 1$ , (94) is true (all terms are defined to be 0 at  $T + 1$ ). Assume (94) is true at  $t + 1$ . Let  $u_t^{prs} = (u_t^0, \bar{u}_t^{1:N}, q_t^{1:N}) = \phi^{prs}(h_t^0)$ . Following an argument similar to that of (95)-(99),

$$\begin{aligned}
& \mathbb{E}^{\phi^{prs}} \left[ \sum_{s=t}^T c_s(X_s^{1:N}, U_s^{0:N}) \middle| h_t^0 \right] \\
& \geq IC(\theta_t^{1:N}, u_t^{prs}) + \sum_{\gamma_{t+1}^1 \in \{0,1\}} \dots \sum_{\gamma_{t+1}^N \in \{0,1\}} \prod_{n=1}^N LS(p^n, \gamma_{t+1}^n) \times \\
& \quad \int V_{t+1} \left( \left\{ NB(\gamma_{t+1}^n, \alpha_t^n, x_{t+1}^n) \right\} \right) \prod_{n \in \mathcal{N}} \alpha_t^n(dx_{t+1}^n) \\
& \geq V_t(\theta_t^{1:N}).
\end{aligned} \tag{100}$$

where the last inequality follows from the definition of the value function (39). This completes the proof of the induction step, and the proof of the theorem.

## APPENDIX F

### PROOF OF THEOREM 2

The proof is done by induction. At  $T + 1$ , (41) is true since  $P_{T+1} = \mathbf{0}$  and  $\tilde{P}_{T+1}^{nn} = \mathbf{0}$  for  $n \in \mathcal{N}$ . Suppose (41) is true at  $t + 1$  and the matrices are all PSD and  $G_{t+1}^{UU}, \tilde{G}_{t+1}^{UU}$  are PD.

At time  $t$ , since  $P_{t+1}$  and consequently  $P_{t+1}^{nn}$  are PSD,  $L_t$  and  $\hat{L}_t^{nn}$  for  $n \in \mathcal{N}$  are PSD. Furthermore, since  $\tilde{P}_{t+1}^{nn}$  is PSD,  $\tilde{L}_t^{nn}$  for  $n \in \mathcal{N}$  is PSD. Since  $R_t^{UU}$  is PD,  $G_t^{UU} = R_t^{UU} + \begin{bmatrix} 0 & I \end{bmatrix} L_t \begin{bmatrix} 0 & I \end{bmatrix}^\top = R_t^{UU} + B^\top P_{t+1} B$  and

$$\begin{aligned}
\tilde{G}_t^{UU} &= R_t^{UU} + \sum_{n=1}^N \begin{bmatrix} 0 & I \end{bmatrix} ((1 - p^n) \hat{L}_t^{nn} + p^n \tilde{L}_t^{nn}) \begin{bmatrix} 0 \\ I \end{bmatrix} \\
&= R_t^{UU} + \sum_{n=1}^N \left( \begin{bmatrix} B \end{bmatrix}_{r,n} \right)^\top ((1 - p^n) P_{t+1}^{nn} + p^n \tilde{P}_{t+1}^{nn}) \begin{bmatrix} B \end{bmatrix}_{r,n}
\end{aligned} \tag{101}$$

are also PD. Then  $P_t$  and consequently  $P_t^{nn}$  for  $n \in \mathcal{N}$  are PSD because  $P_t$  is the Schur complement of  $G_t^{UU}$  of the matrix  $G_t$ . Similarly,  $\tilde{P}_t^{nn}$  for  $n \in \mathcal{N}$  is PSD because  $\tilde{P}_t^{nn}$  is the Schur complement of  $\tilde{G}_t^{UU}$  of the matrix  $\begin{bmatrix} \tilde{G}_t^{X^n X^n} & \tilde{G}_t^{X^n U^n} \\ \tilde{G}_t^{U^n X^n} & \tilde{G}_t^{U^n U^n} \end{bmatrix}$ .

Let's now compute the value function at  $t$  given by (39) in Theorem 1. We first consider the second term of the value function in (39). From the induction hypothesis we have

$$\begin{aligned}
& V_{t+1} \left( \left\{ NB(\gamma_{t+1}^n, \alpha_t^n, x_{t+1}^n) \right\} \right) \\
&= QF \left( P_{t+1}, \mathbf{vec} \left( \left\{ \mu \left( NB(\gamma_{t+1}^n, \alpha_t^n, x_{t+1}^n) \right) \right\} \right) \right) \\
&+ \sum_{n \in \mathcal{N}} \mathbf{tr} \left( \tilde{P}_{t+1}^{nn} \mathbf{cov} \left( NB(\gamma_{t+1}^n, \alpha_t^n, x_{t+1}^n) \right) \right) + e_{t+1} \\
&= QF \left( P_{t+1}, \mathbf{vec} \left( \left\{ \mu(\alpha_t^n) + \gamma_{t+1}^n (x_{t+1}^n - \mu(\alpha_t^n)) \right\} \right) \right) \\
&+ \sum_{n \in \mathcal{N}} (1 - \gamma_{t+1}^n) \mathbf{tr} \left( \tilde{P}_{t+1}^{nn} \mathbf{cov}(\alpha_t^n) \right) + e_{t+1} \tag{102}
\end{aligned}$$

where the last equality in (102) is true because  $NB(\gamma_{t+1}^n, \alpha_t^n, x_{t+1}^n) = (1 - \gamma_{t+1}^n) \alpha_t^n + \gamma_{t+1}^n \rho(x_{t+1}^n)$ ,  $\mu(\rho(x_{t+1}^n)) = x_{t+1}^n$ , and  $\mathbf{cov}(\rho(x_{t+1}^n)) = 0$ .

The first term on the right hand side of (102) can be further decomposed into

$$\begin{aligned}
& QF \left( P_{t+1}, \mathbf{vec} \left( \left\{ \mu(\alpha_t^n) + \gamma_{t+1}^n (x_{t+1}^n - \mu(\alpha_t^n)) \right\} \right) \right) \\
&= QF \left( P_{t+1}, \mathbf{vec} \left( \left\{ \mu(\alpha_t^n) \right\} \right) \right) \\
&+ 2 \mathbf{vec} \left( \left\{ \mu(\alpha_t^n) \right\} \right)^\top P_{t+1} \mathbf{vec} \left( \left\{ \gamma_{t+1}^n (x_{t+1}^n - \mu(\alpha_t^n)) \right\} \right) \\
&+ QF \left( P_{t+1}, \mathbf{vec} \left( \left\{ \gamma_{t+1}^n (x_{t+1}^n - \mu(\alpha_t^n)) \right\} \right) \right). \tag{103}
\end{aligned}$$

Note that  $\int (x_{t+1}^n - \mu(\alpha_t^n)) \alpha_t^n (dx_{t+1}^n) = 0$ ,  $\forall n \in \mathcal{N}$  and

$\int (x_{t+1}^n - \mu(\alpha_t^n)) (x_{t+1}^m - \mu(\alpha_t^m)) \alpha_t^n (dx_{t+1}^n) \alpha_t^m (dx_{t+1}^m) = 0$ ,  $\forall n \neq m$ . Consequently, integrating the right hand side of (102) with respect to  $\prod_{n \in \mathcal{N}} \alpha_t^n (dx_{t+1}^n)$  we get

$$\begin{aligned}
& \int V_{t+1} \left( \left\{ NB(\gamma_{t+1}^n, \alpha_t^n, x_{t+1}^n) \right\} \right) \prod_{n \in \mathcal{N}} \alpha_t^n (dx_{t+1}^n) \\
&= QF \left( P_{t+1}, \mathbf{vec} \left( \left\{ \mu(\alpha_t^n) \right\} \right) \right) + \sum_{n \in \mathcal{N}} \gamma_{t+1}^n \mathbf{tr} \left( P_{t+1}^{nn} \mathbf{cov}(\alpha_t^n) \right) \\
&+ \sum_{n \in \mathcal{N}} (1 - \gamma_{t+1}^n) \mathbf{tr} \left( \tilde{P}_{t+1}^{nn} \mathbf{cov}(\alpha_t^n) \right) + e_{t+1}. \tag{104}
\end{aligned}$$

Substituting (104) back into (39), the second term of the value function in (39) can be written as

$$\begin{aligned}
& QF \left( P_{t+1}, \mathbf{vec} \left( \left\{ \mu(\alpha_t^n) \right\} \right) \right) + \sum_{n \in \mathcal{N}} (1 - p^n) \mathbf{tr} \left( P_{t+1}^{nn} \mathbf{cov}(\alpha_t^n) \right) \\
&+ \sum_{n \in \mathcal{N}} p^n \mathbf{tr} \left( \tilde{P}_{t+1}^{nn} \mathbf{cov}(\alpha_t^n) \right) + e_{t+1}. \tag{105}
\end{aligned}$$

Let  $S_t^{\theta_t} := \text{vec}(\{X_t^{\theta_t}\}, u_t^0, \{\bar{u}_t^n + q_t^n(X_t^{\theta_t})\})$  where  $X_t^{\theta_t}$  is a random vector with distribution  $\theta_t^n$  and  $\{X_t^{\theta_t}\}$  and  $W_t^{1:N}$  are independent. Let  $Y_t^{\theta_t^n}$  be the random vector defined by

$$\begin{aligned} Y_t^{\theta_t^n} &:= \begin{bmatrix} A & B \end{bmatrix}_{r,n} S_t^{\theta_t} + W_t^n \\ &= A^{nn} X_t^{\theta_t^n} + B^{nn}(\bar{u}_t^n + q_t^n(X_t^{\theta_t^n})) + B^{n0} u_t^0 + W_t^n. \end{aligned}$$

From (34) in Lemma 4 we know that  $Y_t^{\theta_t^n}$  has distribution  $\alpha_t^n$  for all  $n \in \mathcal{N}$ . Then, (105) becomes

$$\begin{aligned} & QF\left(P_{t+1}, \text{vec}(\{\mathbb{E}[Y_t^{\theta_t^n}]\})\right) + \sum_{n \in \mathcal{N}} (1-p^n) \text{tr}\left(P_{t+1}^{nn} \text{cov}(Y_t^{\theta_t^n})\right) \\ & + \sum_{n \in \mathcal{N}} p^n \text{tr}\left(\tilde{P}_{t+1}^{nn} \text{cov}(Y_t^{\theta_t^n})\right) + e_{t+1} \\ & = QF\left(L_t, \mathbb{E}[S_t^{\theta_t}]\right) + \sum_{n \in \mathcal{N}} \text{tr}\left(((1-p^n)\hat{L}_t^{nn} + p^n \tilde{L}_t^{nn}) \text{cov}(S_t^{\theta_t})\right) \\ & + \sum_{n \in \mathcal{N}} \text{tr}\left(((1-p^n)P_{t+1}^{nn} + p^n \tilde{P}_{t+1}^{nn}) \text{cov}(\pi_{W_t^n})\right) + e_{t+1} \\ & = QF\left(L_t, \mathbb{E}[S_t^{\theta_t}]\right) \\ & + \sum_{n \in \mathcal{N}} \text{tr}\left(((1-p^n)\hat{L}_t^{nn} + p^n \tilde{L}_t^{nn}) \text{cov}(S_t^{\theta_t})\right) + e_t. \end{aligned} \tag{106}$$

The first equality in (106) is true because  $S_t^{\theta_t}$  and  $W_t^{1:N}$  are independent, and the second equality in (106) follows from the definition of  $e_t$  in (42).

Using the random vector  $S_t^{\theta_t}$ , we can write the first term of the value function in (39) as

$$\mathbb{E}\left[QF\left(R_t, S_t^{\theta_t}\right)\right] = QF\left(R_t, \mathbb{E}\left[S_t^{\theta_t}\right]\right) + \text{tr}\left(R_t \text{cov}(S_t^{\theta_t})\right). \tag{107}$$

Now putting (107) and (106) (that is, the first and second terms of the value function in (39)) together into the value function equation (39) we get

$$\begin{aligned} V_t(\theta_t^{1:N}) &= e_t + \min_{\{q_t^n \in \mathcal{Q}^n(\theta^n)\}} \left\{ \min_{\{\bar{u}_t^n \in \mathbb{R}^{d_U^n}\}, u_t^0 \in \mathbb{R}^{d_U^0}} \left\{ \right. \right. \\ & \quad \left. \left. QF\left(G_t, \mathbb{E}\left[S_t^{\theta_t}\right]\right) + \text{tr}\left(\tilde{G}_t \text{cov}(S_t^{\theta_t})\right) \right\} \right\}. \end{aligned} \tag{108}$$

Note that  $\mathbb{E}[q_t^n(X^{\theta^n})] = 0$  for  $n \in \mathcal{N}$ , and consequently,  $\mathbb{E}[S_t^{\theta_t}] = \text{vec}(\{\mu(\theta_t^n)\}, u_t^0, \bar{u}_t^{1:N})$  depends only on  $u_t^0, \bar{u}_t^{1:N}$ . Furthermore,  $\text{cov}(S_t^{\theta_t}) = \text{cov}(\text{vec}(\{X^{\theta_t^n}\}, 0, \{q_t^n(X^{\theta_t^n})\}))$  depends only on the choice of  $q_t^{1:N}$ . Consequently, the optimization problem in the (39) can be further simplified to be

$$\begin{aligned} V_t(\theta_t^{1:N}) &= e_t + \min_{u_t^0, \bar{u}_t^{1:N}} QF(G_t, \text{vec}(\{\mu(\theta_t^n)\}, u_t^0, \bar{u}_t^{1:N})) \\ &+ \min_{\{q_t^n \in \mathcal{Q}^n(\theta^n)\}} \text{tr}(\tilde{G}_t \text{cov}(\text{vec}(\{X^{\theta_t^n}\}, 0, \{q_t^n(X^{\theta_t^n})\}))). \end{aligned} \quad (109)$$

Now we need to solve the two optimization problems

$$\min_{u_t^0, \bar{u}_t^{1:N}} QF(G_t, \text{vec}(\{\mu(\theta_t^n)\}, u_t^0, \bar{u}_t^{1:N})), \quad (110)$$

$$\min_{\{q_t^n \in \mathcal{Q}^n(\theta^n)\}} \text{tr}(\tilde{G}_t \text{cov}(\text{vec}(\{X^{\theta_t^n}\}, 0, \{q_t^n(X^{\theta_t^n})\}))). \quad (111)$$

Since  $G_t^{UU}$  is PD, it follows by the first part of Claim 3 that the optimal solution of (110) is given by (52) and

$$\begin{aligned} &\min_{u_t^0, \bar{u}_t^{1:N}} QF(G_t, \text{vec}(\{\mu(\theta_t^n)\}, u_t^0, \bar{u}_t^{1:N})) \\ &= QF(P_t, \text{vec}(\{\mu(\theta_t^n)\})). \end{aligned} \quad (112)$$

Similarly, since  $\tilde{G}_t^{UU}$  is also PD, the second part of Claim 3 implies that the optimal solution of (111) is given by (53) and

$$\begin{aligned} &\min_{\{q_t^n \in \mathcal{Q}^n(\theta^n)\}} \text{tr}(\tilde{G}_t \text{cov}(\text{vec}(\{X^{\theta_t^n}\}, 0, \{q_t^n(X^{\theta_t^n})\}))) \\ &= \sum_{n=1}^N \text{tr}(\tilde{P}_t^{nn} \text{cov}(\theta_t^n)). \end{aligned} \quad (113)$$

Finally, substituting (112) and (113) into (109) we obtain (41) at  $t$ . This completes the proof of the induction step and the proof of the theorem.

## APPENDIX G

### PROOF OF THEOREM 3

Let  $\hat{X}_t^n$ ,  $n \in \mathcal{N}$ , be the estimate (conditional expectation) of  $X_t^n$  based on the common information  $H_t^0$ . Then, for any realization of the marginal common belief  $\theta_t^n$ ,  $\hat{x}_t^n = \mu(\theta_t^n)$  for all

$n \in \mathcal{N}$ . This together with Theorems 1 and 2 result in (54) and (55). To show (56) and (57), note that at time  $t = 0$ , for any  $n \in \mathcal{N}$  and for any realization  $h_t^0$  of  $H_t^0$ ,

$$\begin{aligned}\hat{x}_0^n &= \mu(\theta_0^n) = \int y \theta_0^n(dy) \\ &= \begin{cases} \int y \pi_{X_0^n}(dy) = \mu(\pi_{X_0^n}) & \text{if } z_0^n = \emptyset, \\ \int y \mathbb{1}_{\{y\}}(x_0^n)(dy) = x_0^n & \text{if } z_0^n = x_0^n. \end{cases}\end{aligned}\quad (114)$$

Therefore, (56) is true. Furthermore, at time  $t + 1$  and for any realization  $h_{t+1}^0$  of  $H_{t+1}^0$ , let  $\theta_{t+1}^{1:N}$  be the corresponding common beliefs and  $u_t^{prs*} = \phi_t^{prs*}(h_t^0)$ , then

$$\hat{x}_{t+1}^n = \mu(\theta_{t+1}^n) = \int y \psi_t^n(\theta_t^n, u_t^{prs*}, z_{t+1}^n)(dy).$$

If  $z_{t+1}^n = x_{t+1}^n$ , then  $\hat{x}_{t+1}^n = \int y \mathbb{1}_{\{y\}}(x_{t+1}^n)(dy) = x_{t+1}^n$ .

If  $z_{t+1}^n = \emptyset$ , then,

$$\begin{aligned}\hat{x}_{t+1}^n &= \int y \psi_t^n(\theta_t^n, u_t^{prs*}, q_t^n, \emptyset)(dy) \\ &= \int y \int \int \mathbb{1}_{\{y\}}(f_t^n(x_t^n, w_t^n, u_t^{prs*})) \theta_t^n(dx_t^n) \pi_{W_t^n}(dw_t^n)(dy) \\ &= \int \int f_t^n(x_t^n, w_t^n, u_t^{prs*}) \theta_t^n(dx_t^n) \pi_{W_t^n}(dw_t^n) \\ &= A^{nn} \hat{x}_t^n + B^{nn} \bar{u}_t^{n*} + B^{n0} u_t^{0*}.\end{aligned}\quad (115)$$

where the third equality is true because

$$\int y \mathbb{1}_{\{y\}}(f_t^n(x_t^n, w_t^n, u_t^{prs*})) dy = f_t^n(x_t^n, w_t^n, u_t^{prs*}).$$

Furthermore, the last equality is true because  $q_t^n \in \mathcal{Q}^n(\theta^n)$  and  $W_t^n$  is a zero mean random vector. Therefore, (57) is true and the proof is complete.